PROPAGATION OF ION ACOUSTIC SOLITON AROUND THE CRITICAL VALUES OF ANY SPECIFIC PARAMETER IN UNMAGNETIZED COLLISIONLESS RELATIVISTIC PLASMAS

By

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Declaration

I hereby declare that the work contained in this Thesis has not been previously submitted to meet requirements for an award at this or any other higher education institution. To the best of my knowledge and belief, the Thesis contains no material previously published or written by another person except where due reference is cited. Furthermore, the Thesis complies with PLAGIARISM and ACADEMIC INTEGRITY regulation of CUET.

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Dedication

"Dedicated To My Family."

List of Publications

[1] Md. Obaidur Rahman et al., "Propagation of ion acoustic solitons around the critical values in weakly relativistic unmagnetized plasmas having non-thermal distributed electrons and positrons," *Proceedings of the 2nd International Conference on Nonlinear Dynamics and Applications (ICNDA 2024)*, Springer Proceedings in Physics, Volume 1, (2024), DOI: 10.1007/978-3-031-66874-6_3.

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[3] Md. Obaidur Rahman et al., "Propagation of ion acoustic solitons with dynamical behaviors around the supercritical values in relativistic unmagnetized plasmas ," under submission.

Approval/Declaration by the Supervisor(s)

This is to certify that Md. Obaidur Rahman has carried out this research work under our supervision, and that he has fulfilled the relevant Academic ordinance of the Chittagong University of Engineering & Technology, so that he is qualified to submit the following thesis in the application for the degree of MASTER OF PHILOSOPHY IN MATHEMATICS. Furthermore, the Thesis complies with PLAGIARISM and ACADEMIC INTEGRITY regulation of CUET.

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Abstract

The main purpose of the present work is to investigate how electrostatic plasma parameters modify the nonlinear ion acoustic soliton (IAS) propagation in unmagnetized collisionless plasma including higher order Lorentz relativistic expansion terms. The study of IASs in an unmagnetized collisionless relativistic plasma made of relativistic ion fluids, Cairns-distributed electrons and Cairns-distributed positrons. In one dimensional analysis, the reductive perturbation technique is employed to reduce the dynamics of the whole system to the Korteweg-de Vries equation (KdVE) involving various nonlinearity, whose nonlinear and dispersion coefficients are dependent on the related plasma parameters. This indicates that KdVEs have been derived incorporating quadratic, cubic, and quartic nonlinearities. However, as the coefficient of KdVE associated with quadratic nonlinearity approaches zero, the method encounters limitations. To overcome this challenge, adjustments are made to the stretching coordinates, resulting in a cubic nonlinearity KdVE that effectively describes soliton propagation near critical values in these plasma conditions. Additionally, a KdVE with quartic nonlinearity is derived to model super-critical values of specific plasma parameters in relativistic plasmas.

Previous studies have primarily focused on relativistic effects on soliton propagation using Lorentz relativistic factor expansions up to three terms. In contrast, this thesis expands this consideration to more higher order Lorentz relativistic expansion terms to minimize truncation errors in modeling nonlinear soliton propagation within these plasmas. The investigation reveals that the relativistic streaming factor significantly alters the wave potential functions with the presence of more higher order Lorentz relativistic expansion terms. Notably, the derived KdVE shows that quadratic nonlinearity supports both compressive and rarefactive soliton propagation, whereas cubic and quartic nonlinearities exclusively support compressive solitons. Furthermore, this study explores how plasma parameters, incorporating more higher order Lorentz relativistic expansion terms, influence the amplitude and width of IASs in the unmagnetized relativistic plasma. It finds that higher order terms of the Lorentz relativistic factor noticeably modify the propagation characteristics of IASs

within this specific plasma environment. The effect of plasma parameters on the amplitude and width of IASs has also been discussed with the physical interpretations.

বিমূর্ত

বর্তমান কাজের মূল উদ্দেশ্য হল ইলেক্ট্রোস্ট্যাটিক প্লাজমা প্যারামিটারগুলি কীভাবে উচ্চ-ক্রমের লরেঞ্জ আপেক্ষিক সম্প্রসারণ পদগুলি সহ অচুম্বকীয় সংঘর্ষহীন প্লাজমায় অরৈখিক Ion Acoustic Soliton (IAS) এর প্রসারণকে সংশোধন করে তা তদন্ত করা। আপেক্ষিক আয়ন তরল, Cairns-বন্টিত ইলেক্ট্রন এবং Cairns-বন্টিত পজিট্রন দিয়ে তৈরি একটি অনির্বাচিত সংঘর্ষহীন আপেক্ষিক প্লাজমায় IAS-এর অধ্যয়ন। এক মাত্রিক বিশ্লেষণে, হ্রাসকারী বিশৃঙ্খলা কৌশলটি সমগ্র সিস্টেমের গতিশীলতাকে Korteweg-de Vries সমীকরণে (KdV) হ্রাস করার জন্য নিযুক্ত করা হয়, যা বিভিন্ন অরৈখিকতার সাথে জড়িত, যার অরৈখিক এবং বিচ্ছুরণ সহগগুলি সম্পর্কিত প্লাজমা প্যারামিটারগুলির উপর নির্ভরশীল। এটি নির্দেশ করে যে, KdV সমীকরণগুলি দ্বিঘাত, ত্রিঘাত এবং চতুর্ঘাত অরৈখিকতাকে অন্তর্ভুক্ত করে উদ্‌ভূত হয়েছে। যাইহোক, দ্বিঘাত অরৈখিকতার সাথে যুক্ত KdV সমীকরণের সহগ শূন্যের দিকে এগিয়ে যাওয়ার সাথে সাথে পদ্ধতিটি সীমাবদ্ধতার মুখোমুখি হয়। এই চ্যালেঞ্জটি কাটিয়ে উঠতে, প্রসারিত স্থানাঙ্কে সমন্বয় করা হয়, যার ফলে একটি ত্রিঘাত অরৈখিক KdV সমীকরণ গঠিত হয় যা কার্যকরভাবে এই প্লাজমা অবস্থার সংকট মানগুলির কাছাকাছি Soliton এর বিস্তারকে বর্ণনা করে। উপরন্তু, চতুর্ঘাত অরৈখিকতা সহ একটি KdV সমীকরণ আপেক্ষিক প্লাজমাতে নির্দিষ্ট প্লাজমা প্যারামিটারগুলির Super-critical মানগুলির মডেল তৈরি করতে উদ্‌ভূত হয়। পূর্ববর্তী গবেষণাগুলি প্রাথমিকভাবে তিনটি পদ পর্যন্ত লরেঞ্জ আপেক্ষিক ফ্যাক্টর সম্প্রসারণ ব্যবহার করে Soliton এর প্রসারণের উপর আপেক্ষিক প্রভাবের উপর দৃষ্টি নিবদ্ধ করেছে। বিপরীতে, এই থিসিসটি এই প্লাজমাগুলির মধ্যে অরৈখিক Soliton প্রসারণের মডেলিংয়ে ছাঁটাই ত্রুটিগুলি হ্রাস করতে আরও উচ্চতর ক্রমের লরেঞ্জ আপেক্ষিক সম্প্রসারণ পদগুলিতে এই বিবেচনাকে প্রসারিত করে। তদন্তটি প্রকাশ করে যে আপেক্ষিক স্ট্রিমিং ফ্যাক্টরটি আরও উচ্চতর ক্রমের লরেঞ্জ আপেক্ষিক সম্প্রসারণ পদগুলির উপস্থিতির সাথে তরঙ্গের সম্ভাব্য ফাংশনগুলিকে উল্লেখযোগ্যভাবে পরিবর্তন করে। উল্লেখ্য যে, উদ্‌ভূত KdV সমীকরণ দেখায় যে দ্বিঘাত অরৈখিকতা সংকোচক এবং বিরল সক্রিয় Soliton প্রসারণ উভয়কেই সমর্থন করে, যেখানে ত্রিঘাত এবং চতুর্ঘাত অরৈখিকতা একচেটিয়াভাবে সংকোচক Solitonকে সমর্থন করে। উপরন্তু, এই গবেষণাটি অনুসন্ধান করে যে কীভাবে প্লাজমা প্যারামিটারগুলি, আরও উচ্চতর ক্রমের লরেঞ্জ আপেক্ষিক সম্প্রসারণ পদগুলিকে অন্তর্ভুক্ত করে, অচুম্বকীয় আপেক্ষিক প্লাজমাতে IAS এর প্রশস্ততা এবং প্রস্থকে প্রভাবিত করে। এতে দেখা গেছে যে লরেঞ্জ আপেক্ষিক ফ্যাক্টরের উচ্চক্রমের পদগুলি এই নির্দিষ্ট প্লাজমা পরিবেশের মধ্যে IAS এর প্রসারণ বৈশিষ্ট্যগুলিকে লক্ষণীয়ভাবে পরিবর্তন করে। IAS এর বিস্তার এবং প্রস্থের উপর প্লাজমা প্যারামিটারগুলির প্রভাবও ভৌত ব্যাখ্যার সাথে আলোচনা করা হয়েছে।

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Nomenclature

KdVE	Korteweg-de Vries Equation
mKdVE	Modified Korteweg-de Vries Equation
RPT	Reductive Perturbation Technique
LRF	Lorentz Relativistic Factor
IAW	Ion Acoustic Wave
IAS	Ion Acoustic Soliton
e-p-i	electron-positron-ion
e-p	electron-positron
CV	Critical value
SCV	Super-critical value
UMRP	Unmagnetized relativistic plasma

Chapter 1: INTRODUCTION

1.1 DEFINITION OF PLASMA

Around 15 billion years ago, the universe was unstable and burst into a gigantic explosion when it was compressed into a tiny ball. The matter generated by this explosion was incredibly hot, causing everything to exist in the form of plasma. Thus, plasma was the initial state of matter at the very inception of the Universe. While expanding the Universe, the temperature of the matter decreased and some of the plasma became gas and further reduction of temperature converted to gas and finally to the solid state. This is the opposite order of events as we know that the solid state is the first state of matter and sequentially the plasma is regarded as the fourth state of matter. At the early stages of civilization, man utilized land, water, and the rain. Usually they recognized the land and rocks as solid and the water as liquid state of matter and nowadays the solid and the liquid are referred as the first and second state of matter, respectively. The existence of the third state of matter was established by the English physicist Robert Boyle discovered the first physical law of gases in 1662. However, the existence of plasma was introduced only about a century ago. It is different from the plasma of blood which is a transparent liquid obtained after removing of its various corpuscles from blood. Such plasma was named by the Czech medical scientist Johannes Purkinje after the Greek word $'\pi\lambda\alpha\sigma\mu\alpha'$ which means "moldable substance" or "jelly". Blood plasma transports red and white blood cells akin to how an electrified fluid conveys electrons and ions. Using this concept, the American Nobel laureate Irving Langmuir described an ionized gas and named the new state of matter "Plasma". Actually, plasma is considered as a gas that is sufficiently ionized to show plasma-like behavior as any ionized gas cannot be called plasma. It is to be noted that plasma-like behavior occurs only after a significant fraction of the atoms of a gas have experienced ionization . Due to the ionization of neutral atoms, plasma is produced and it has almost equal number of positive and negative charge carriers. In this case, the opposite charges are tightly associated and tend to neutralize one another on a macroscopically large scale. Therefore, plasma is defined as a quasineutral gas of charged and neutral particles and it reveals collective behavior [2–4]. So, plasma is an ionized gas containing charged particles, and neutral atoms with many degrees of ionization. Although plasma does not exist on Earth, it is found in almost all interstellar and extra-galactic objects. Gaseous nebulae, hot stars and upper atmosphere like Ionosphere are such kind of objects. Also flames (i.e. fire), lightning and aurorae contain plasma. Since more than 99% of the matter in visible space is supposed to be in the plasma state, the Earth magnetosphere, the Van Allen radiation belts, solar wind, solar corona and the core of the sun are filled with plasma. Point to be noted that where we live on the earth, only 1% of the universe, plasma does not occur here naturally. However, it can be generated in laboratories via electric discharge, photoionization, heating gas with sufficiently large temperature etc. Plasmas in fusion energy research, plasma globes, laser produced plasmas, rocket exhaust and ion thrusters, inside fluorescent lamps and neon signs, plasma tv etc. are some examples of artificial and laboratory plasma.

1.2 PROPERTIES AND BASIC CRITERIA

The force resulting from electric interaction between charged particles is governed by Coulomb's law. This law states that opposite charges attract each other (e.g., protons attract electrons), while like charges repel each other.

However, in plasma, there is a Coulomb force between the charged particles. Let us suppose an ion in plasma which possesses a positive charge. Due to the Coulomb force, a huge number of electrons gets attracted and surrounds the positive ion like a cloud. The nearest electrons of the ion build a shield for the rest of existing electrons so that the force between the ion and the shielded electrons is smaller than the Coulomb force without shielding. The electrons far from the ion are also shielded but the force decreases successively in every layer. Thus, the attraction force of a positive ion is not approach to infinity but to a finite distance [5]. This finite distance is defined as the Debye radius or Debye length and such a shielding is called the Debye shielding. Since a plasma is a composition of negatively and positively charged particles, i.e. electrons and ions, the positively charged ions almost completely neutralize the negatively charged electrons. Such an incident in plasma is defined as quasineutrality [6]. Thus, in a quasineutral mixture of charged particles e.g. plasma, the densities of positively and negatively charged particles are almost the same. However, quasineutrality hold only in the macroscopic scale. The first characteristic of an ionized gas to be plasma is that it must hold the quasineutrality condition. Quasineutrality is obtained if the characteristic dimension L of the plasma system is greater than the Debye length λ_D. Thus, the first criterion of plasma [2,4,7] is as follows:

$$L \gg \lambda_D, \tag{1.1}$$

Since a plasma exhibits collective behavior inside a Debye sphere, the number density of electrons must be enormously large. So, the second criterion for being plasma is

$$N_D \approx n_e \lambda_D^3 \gg 1,$$

Another criterion for a gas to behave like a plasma is

$$\omega_p \tau \gg 1, \tag{1.2}$$

where ω_p is the frequency of plasma and τ is the mean time between collisions in plasma particles.

1.3 TYPES OF PLASMA

Plasmas have many characteristics based on degrees of ionization, temperature, density and effect of the magnetic field. So, there are many types of plasma in the universe ranging from very high density to low density. Fig. 1.1 demonstrates several plasmas as a function of density and temperature. Plasma is also classified into several types. Some of the varieties of plasmas are discussed below.

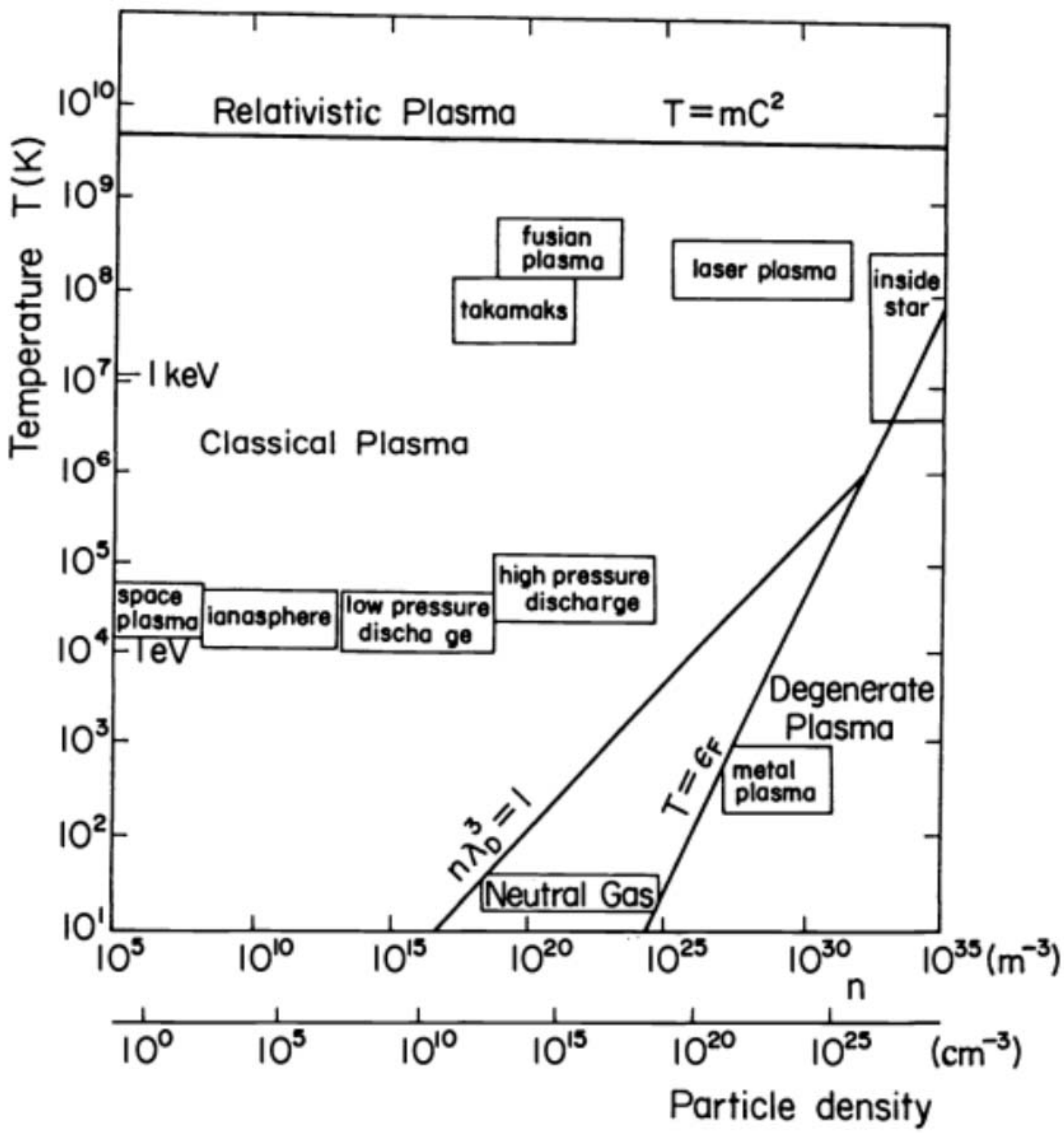

Figure 1.1: Various types of plasmas in a logarithmic temperature-density parameter space [1].

1.3.1 THERMAL AND NON-THERMAL PLASMA

A plasma contains electrons, ions and neutral atoms with different temperatures. According to the comparative temperatures of the particles, plasma may be classified into thermal and non-thermal plasma. A plasma is said to be thermal plasma if all the neutral atoms are almost fully ionized and there is a thermal equilibrium in the particles since the temperatures of the electrons and ions are almost equal i.e. $T_e \approx T_i$ ($T_e =$ electron temperature and $T_i =$ ion temperature). Sometimes, thermal plasma is called hot plasma. However, thermal plasmas are produced in two different environments such as when the heavy ions possess extremely high thermal energy i.e. $10^2 - 10^4$ eV at the temperatures in the order of $10^6 - 10^8$ k. Besides, if the atmospheric pressure is large, thermal plasmas are generated in the low temperature even at 6000 k. On the other hand, non-thermal plasma is such a plasma where a tiny portion of total number of atoms or molecules is ionized. Like thermal plasma, non-thermal plasmas also exist in two situations. When the electron temperature enormously greater than the temperature of the heavy ions i.e. in the order of $10^4 - 10^5\, k\, (1 - 10\, eV)$, then cold plasma is generated. Cold plasmas may also be produced at low temperature as room temperature [8].

1.3.2 MAGNETIZED AND UNMAGNETIZED PLASMA

Plasma system may be significantly influenced by the magnetic field. In accordance with the presence of magnetic field, plasmas can be categorized into magnetized and unmagnetized plasma. When the velocities of charged particles of a plasma are remarkably dominated by a strong magnetic field, then the plasma is said to be magnetized plasma. As plasma is a good conductor, so the electric field is weak. While moving the charged particles of plasma in a magnetic field, the Debye shielding does not affect the electric field. The electric field associated with the magnetic field is expressed as

$$\vec{E} = \vec{v} \times \vec{B}, \tag{1.3}$$

where $\vec{E}$, $\vec{v}$ and $\vec{B}$ indicate the electric field, mean velocity of plasma particles and magnetic field, respectively. The charged particles are governed in such a field by the Lorentz force as

$$\vec{F} = q\left(\vec{E} + \vec{B}\right), \tag{1.4}$$

where q is the charge of particle. On the contrary, a plasma is said to be unmagnetized if there is no magnetic field applied in the plasma system or due to the motion of the charged particles, there is a negligible instinctive magnetic field i.e. $\vec{B} \to 0$. A uniform electric field dominates the motion of the charged particles in such kind of plasma as the force is

$$\vec{F} = q\vec{E}. \tag{1.5}$$

1.3.3 COLLISIONAL AND COLLISIONLESS PLASMA

In the kinetic theory of gases, the mean free path (average distance between two collisions) plays a vital role in the motion of particles. Rely on the mean free path and interaction among the particles, plasma can be collisional or collisionless. When the dimension of the plasma system (L) is sufficiently larger than the mean free path (λ_{mp}) for binary Coulomb collision i.e.$(L \gg \lambda_{mp})$, then the plasma is categorized as collisional plasma [3,8]. Whereas, the collisionless plasmas are plasma environments where the interactions between individual particles occur primarily through electromagnetic forces rather than collisions. In these plasmas, particles such as electrons and ions move freely and do not collide frequently with each other. This unique characteristic arises when the mean free path for collisions between particles becomes larger than the typical spatial scales of the plasma, or when the collision frequency becomes very low. Understanding collisionless plasmas is crucial for studying phenomena in space physics, astrophysics, and in developing technologies such as fusion energy research and advanced plasma-based devices.

1.4 MULTI-SPECIES RELATIVISTIC PLASMA

Electrons along with positrons whereas they possess equal mass and opposite charges constitute electron-positron (e-p) plasma. Most of the cosmic objects are composed with e-p plasmas. Active galactic nuclei, pulsar magnetospheres, Van Allen radiation belts, the solar atmosphere are the reservoir of such plasmas [9–12]. It is assumed that e-p plasmas may exist in the upper regions of the magnetospheres of rotating neutron stars due to their extreme electromagnetic radiation [13]. Despite the astrophysical plasmas consisting of e-p plasma, there may exist a small number of heavy ions. With the presence of heavy ions like protons in astrophysical environments, the multi-component involving electrons, positrons and ions plasma may be produced in nature as well as in space. In scenarios like the MeV epoch of the early universe, where temperatures range from $(T \sim 1 - 10^4 MeV)$ up to one second after the Big Bang, the Universe was primarily composed of relativistic e-p plasmas in equilibrium with photons, neutrinos, and antineutrinos [14]. When the plasma reaches temperatures exceeding the rest mass energy of electrons i.e., $m_0c^2 = 0.5MeV$, it becomes relativistic, with e-p pair creation and annihilation $(\gamma = e^+ + e^-)$ processes playing crucial roles. These processes occur over longer timescales compared to the collective interactions between charged particles [15]. In astrophysical phenomena like active galactic nuclei [16], plasma temperatures near black holes can reach $10^7 MeV$ for ions and $10^3 MeV$ for electrons due to rapid cooling. Various studies [17–19] have explored different types of plasmas, including isothermal, cold, and hot plasmas around black holes. Researchers [20, 21] have highlighted the significance of e-p pair creation or annihilation in relativistic plasmas, which can occur during intense interactions such as those with laser pulses. Additionally, plasma interactions with highly energetic cosmic rays in Earth's magnetosphere lead to the production of e-p pairs [22, 23]. The e-p plasmas also exist in environments like the Earth's magneto-tail and outflows from pulsars, interacting with low density electron-ion plasmas [24].

Additionally, positrons with sufficient lifetime can be utilized to investigate particle transport within tokamaks [25, 26]. The plasma undergoes significant modifications when positrons are introduced into electron-ion (e-i) plasmas, transforming the two-component e-i plasma into multi-species plasma. Moreover, many astrophysical compact objects contain e-p-i plasmas, prompting numerous studies [27–30] primarily focused on non-relativistic regimes. However, when plasma particle velocities approach the speed of light, relativistic effects become crucial for studying these plasmas, supported by abundant evidence [31–33] of astrophysical particles moving at relativistic velocities. In environments like the solar atmosphere and interstellar spaces, energetic streaming ions with energies ranging from 0.1 to 100 MeV are frequently observed [31–33]. Therefore, investigating the nonlinear propagation of ion acoustic waves (IAWs) in relativistic e-p-i plasmas has become essential for understanding both astrophysical and laboratory plasma physics. Furthermore, high-energy particles are produced by nonthermal/superthermal particles with energies exceeding thermal energies [34], leading to long-range interactions characterized by non-Maxwellian distributions such as Schamel [35], nonextensive [36,37], Cairns [38], and kappa distributions rather than the Boltzmann distribution. However, recreating astrophysical or space-like plasmas in laboratory settings remains challenging for researchers seeking qualitative insights into fundamental plasma properties. Additionally, nonlinear collective behaviors frequently encountered in plasmas necessitate rigorous mathematical methods for proper study. Addressing these complexities requires appropriate plasma assumptions to elucidate the physical phenomena observed in space and astrophysical contexts. This work aims to derive fluid model equations that capture wave propagation characteristics in multi-species plasmas, encompassing the relativistic regimes.

1.5 NONLINEARITY IN PLASMA

It is well confirmed that the natural systems respond predictably to specific conditions, with small changes in conditions causing small changes in the response. This appealing idea has led to a view of the world as linear, where effects are directly proportional to their causes. Though interesting and reassuring, this long-standing idea is now being challenged. It is seen as only partially accurate, as many situations, especially those affecting our daily lives, show drastic deviations from proportional responses. A key difference between linear and nonlinear laws is whether superposition applies. In a linear system, the combined effect of two causes is just the sum of their individual effects. In a nonlinear system, combining two actions can produce dramatic new effects due to interactions between elements. This can lead to unexpected structures, abrupt transitions, multiple states, pattern formation, or unpredictable changes known as deterministic chaos. Thus, nonlinear science deals with evolution and complexity [39].

Today, research in physics heavily focuses on nonlinear phenomena, as most physical phenomena are inherently nonlinear. Nonlinear science, similar to quantum mechanics and relativity, introduces fundamentally new ideas and surprising results. To address the mathematical challenges of nonlinearity, physics often uses approximations to transform unsolvable nonlinear problems into solvable linear ones [40,41]. Very recently, a significant advance in the study of nonlinear effects, which holds a special place in plasma physics, has been reported [42].

A plasma is inherently a nonlinear medium. Collective processes are crucial in plasma, especially in causing different plasma instabilities. These instabilities often result in an increase in electric field strength, which can become quite significant. The presence of various instabilities is a defining characteristic of plasma as a state of matter. In the linear approximation, various instabilities are studied assuming small perturbation amplitudes, indicating whether waves grow or dampen. However, as the amplitude increases, the linear approximation becomes inadequate. Nonlinear effects typically serve to constrain the growth of instabilities through mechanisms known as nonlinear saturations. High amplitudes lead to nonlinear conversion, potentially causing additional radiation and affecting plasma cooling. Plasma radiation provides vital information about interstellar and solar processes. Nonlinear conversion helps interpret phenomena like solar flares and supernova emissions, offering insights into cosmic ray origins. This also applies to the investigation of the most interesting astronomical objects, namely the quasars discovered in 1962 [43]. They are plasma formations emitting radiation due to nonlinear conversion of plasma waves into transverse waves [44, 45]. This conversion is also observed in solar flares, where plasma emission results from the nonlinear interaction of two plasma waves [46]. Experimental observations of plasma emission through charged particle beam transmission further support this phenomenon [47–50]. Additionally, when external forces excite large amplitude waves, nonlinear effects become significant. For instance, in ion acoustic waves, nonlinearities counteract dispersion, allowing ion acoustic solitons to propagate without significant distortion.

Plasma nonlinear effects vary greatly and can be classified based on electromagnetic field strength, interaction timescales, and characteristic lengths. Distinguishing between weak and strong nonlinearities is crucial. Weak nonlinearities involve describing processes with the first terms of the field's amplitude expansion. Typically, plasma fields are time-varying, with characteristic frequencies ω assigned to them. Nonlinear interactions occurring much faster than collision times are most effective and termed nondissipative. Conversely, when interactions are weak and time far exceeds length over velocity, they become dissipative. In dissipative nonlinearities, the effective collision frequency depends on field amplitude. Dissipative nonlinearities were initially explored in relation to problems of radio wave propagation and have been extensively examined by Ginzburg and Gurevich [51]. It is crucial to emphasize that nonlinear interaction processes undergo substantial changes when the characteristic lengths of certain interacting waves exceed the typical dimensions of the system, particularly in the context of transverse modes. Thus, nonlinearities play a crucial role in the localization of waves, giving rise to various fascinating coherent structures.

1.6 WAVE PHENOMENA IN PLASMA

Production and recombination of interacting charged particles and fields give rise to plasma waves which propagate periodically [8]. Along with the waves, wave-pushed transport, instabilities, fluctuations etc. occur in the plasma system. Plasmas are capable of supporting nonlinear waves due to their inherent complexity. Nonlinear effects arise when nonlinearity interacts with the dispersion or dissipation characteristics of the medium, creating wavelike disturbances that propagate through the plasma. When absorption is present, these waves gradually lose amplitude as they travel. In dispersive media, where different frequencies propagate at different speeds, waves tend to spread out, leading to a decrease in their amplitude over time.

Studying nonlinear waves and oscillations in plasmas is crucial because these wave phenomena establish a close relationship between theoretical predictions and experimental observations. Understanding these interactions is essential for advancing both theoretical models and experimental techniques in plasma physics. Plasma exhibits a diverse array of interconnected wave phenomena involving particles and fields that propagate in periodic patterns. Plasma is typically an electrically conductive quasi-neutral fluid composed of electrons and ions (which can include multiple ion species, negative ions, dust particles, neutral particles, etc.), as well as positrons in some cases. These constituents interact through collective behaviors, coupling via electric and magnetic fields within the plasma medium. This complex interplay underlies the rich variety of wave behaviors observed in plasmas across different scales and conditions. Due to its many degrees of freedom, plasma supports various types of acoustic wave phenomena. These include ion acoustic, electron acoustic, and others, reflecting the diverse interactions and dynamics within the plasma medium.

1.7 ION ACOUSTIC WAVE

Ion acoustic waves (IAWs) are electrostatic waves similar to acoustic waves, where the restoring force originates from the pressure of lighter particles (like electrons, positrons, etc.), and the inertia is due to the ion species' mass density. These waves travel longitudinally and arise from compressions and rarefactions within the plasma medium, akin to sound waves in neutral gases. Ions fulfill a role analogous to neutral atoms in ordinary sound waves. However, IAWs can propagate in collisionless media because ions interact with electrostatic or electromagnetic fields over long distances, unlike sound waves. Additionally, plasmas contain electrons, which influence wave dispersion. Due to electrons' high mobility relative to ions, they quickly adjust to ion motions to maintain charge neutrality. Electron motion results from small internally generated electric fields due to local variations in ion density within the plasma. In the absence of a magnetic field and considering the motions of massive ions, the low-frequency IA mode is excited with a specific dispersion relation

$$\omega^2 = k^2\left[\frac{k_B T_e}{m_i} \times \frac{1}{1+k^2\lambda_{di}^2} + \frac{\gamma_i k_B T_i}{m_i}\right],$$

where ω, k, k_B, λ_{di} and $T_e(T_i)$ are the frequency, propagation constant, Debye's length and electron (ion) temperature, respectively. For $k^2\lambda_{di}^2 >> 1$ and $T_i \to 0$, one obtains

$$\omega^2 = k^2\frac{n_o e^2}{\varepsilon_0 m_i} = \omega_{pi}^2.$$

where ω_{pi} is the ion plasma frequency. Baumjohann and Treumann [52] also derived the dispersion relation considering the motion of both electrons and ions, expressed as

$$\frac{\omega}{k} = \sqrt{\frac{\gamma_e k_B T_e}{m_i}} = v_s,$$

where v_s is the ion acoustic speed. It has been observed that the group velocity of IAWs equals the phase velocity, and these waves are present only when thermal motions of charged particles occur. In IAWs, ions oscillate despite their significant inertia, supported by restoring forces from the pressure exerted by lighter species. In space plasmas, IAWs are often observed as highly energetic particles stream upstream of planetary bow shock fronts. Various types of IAWs exist in plasmas, including solitary waves, shock waves, double layers, and others. These wave phenomena are crucial for advancing our understanding of plasma physics and related phenomena.

1.8 ION ACOUSTIC SOLITON

A soliton is a nonlinear wave characterized by a self-sustaining hump or dip shape that maintains its form and velocity as it propagates. It represents a relatively stable disturbance, which can be seen as a combination of sinusoidal wave trains of varying frequencies. In a non-dispersive medium, where each of these component waves travels at the same velocity, pulses propagate without distortion [40]. Conversely, in a dispersive medium where velocities differ, pulses spread over time. Solitons arise from a balance between nonlinearity and dispersion, with minimal dissipation. Understanding soliton propagation is crucial for comprehending how particles or energy move in plasmas, playing a significant role in nonlinear plasma physics research. High-intensity laser energy can excite nonlinear plasma waves, transferring energy to the plasma. Solitons in plasmas manifest as localized wave modes, such as electrostatic or electromagnetic solitons coupled nonlinearly to space charge fields. These stable, stationary structures result from the interplay between dispersion and nonlinear effects in the medium. Ion acoustic soliton (IAS) propagation can be studied by deriving the Korteweg-de Vries equations (KdVEs) from the proposed , which are mainly a type of localized solitary wave with small but finite amplitudes. These nonlinear wave phenomena, occurring far from thermodynamic equilibrium, can be replicated in laboratory settings under controlled conditions to investigate physical phenomena akin to those observed in astrophysical and space plasma environments [40].

1.9 CAIRNS VELOCITY DISTRIBUTION

The plasma particles are distributed in physical and velocity space. In order to describe the distribution of these particles the distribution functions $f(x,v,t)$ are introduced. When the temperatures of electrons are the same as the ions in the plasma system, then the system is said to be at thermal equilibrium. At the thermal equilibrium, plasma particles move with an average speed. Like temperature, the other physical properties of particles yield average values at the equilibrium position. With the study of the distribution function for such a plasma system, the average values of the physical properties of the particles can be determined. The Maxwell-Boltzmann function is an equilibrium distribution function which represents the most probable distribution suitable to the macroscopic properties of particles at thermal equilibrium [40, 53]. The normalized distribution function for the particle velocities [3] is defined as:

$$f^3(v)d^3vd^3x = n/(\sqrt{(}\pi)v_t)^3 \exp(-v^2/v_t^2), \tag{1.6}$$

where $v_t = \sqrt{(}2K_B\ T_e/m_e)$ is the root mean square thermal speed. However, due to the long-ranged interactions i.e. Coulomb forces and gravitational force, the values of particle properties may deviate from the average values. In such case, the Maxwellian distribution function is inadequate for describing the properties of particles. In most of the astrophysical environments [54–58], plasma properties do not follow the Maxwellian distribution as the particles may not be in thermal equilibrium. Non-Maxwellian distribution is then applied for such types of plasmas. There are many distribution functions to be used as non-Maxwellian cases such as Cairns distribution [38], non-extensive q-distribution [59], kappa distribution [35], generalized $(\alpha, q)-$distribution [38, 59, 60].

Whereas Cairns distribution is assigned in this research work, only the mentioned distribution function is described in the following section. Since the electrons may be isothermal, nonthermal, subthermal or superthermal in plasma system, the Cairns velocity distribution functions are suitable for describing the velocity and energy of electrons. The Cairns velocity distribution function is given by [38]

$$f(v) = C_1 + \alpha(\frac{v}{v_t})^4 \exp(-v^2/2v_t^2), \tag{1.7}$$

where $v_t = \sqrt{(}K_B\ T_e/m_e\ )$ indicates the thermal speed of electron. Here α is the number of nonthermal populations electrons, and $C_1 = N_{e0}/((3\alpha+1)\sqrt{(}2\pi v_t^2\ ))$ is the normalizing constant. Integrating Eq. (1.7) over the total velocity space the electron density (N_e) is obtained as

$$N_e = N_{e0}\ [1 - \Gamma_e\left(\frac{e\psi}{K_BT_e}\right) + \Gamma_e\left(\frac{e\psi}{K_BT_e}\right)^2] \exp\left(\frac{-e\psi}{K_BT_e}\right), \tag{1.8}$$

where $\Gamma_e = (4\alpha_e)/(1+3\alpha_e\ )$ calculates the deviation from the thermalized state and α_e measures the concentration of nonthermal electrons inside the plasma. ψ, T_e and e are the electrostatic potential, electron temperature and absolute value of electric charge, respectively.

1.10 STATEMENT OF THE PROBLEMS

Plasma is a medium of charged particles that follows the collective behavior. There are different types of plasma depending on the degree of ionization and temperature. Plasma can be either fully ionized when all the particles are ionized or partially ionized when the fraction of the neutrals is ionized. This means that the charged particles are produced by different ionization or excitation mechanisms of the neutral particles, which create ions and electrons. Plasma is then a collection of the various charged particles that are free to move and as a whole electrically neutral. However, astrophysical and space plasmas are consistent with electrons, positrons, and ions along with neutrals. The positrons have equal masses but different charges in ordinary electron-ion plasmas. They can be produced in laboratory [61–63] and available in nature such as in the active galactic nuclei [64], in the neutron stars [58], in pulsar magnetospheres [65], at the center of the Milky Way Galaxy [66] and in laser-plasma experiments [67]. Most of the astrophysical and space environments contain ions as well as electrons and positrons, forming multi-species plasmas [68–70]. On the other hand, in plasma, when the velocities of the particles approach or exceed the speed of light, it is referred to as relativistic plasma [27, 32, 33, 71]. Relativistic plasmas have garnered significant attention from researchers due to their diverse applications and potential for investigating various collective processes in astrophysical, space, and laboratory plasmas. They exist not only in the early universe's evolution but also in the inner regions of accretion discs near black holes [27], in the plasma sheath boundary layer of Earth's magnetosphere [27], and in laser-plasma interactions [72], among others.

Such plasmas, characterized by relativistic kinetic energies and arbitrary concentrations, are known to convert into radiation in gamma-ray bursts, though the exact conversion mechanism remains unknown. Recent experimental verifications [73, 74] have addressed some astrophysical issues. Additionally, interactions of relativistic shells with background plasma in shock waves and pulsar wind nebulae have been studied. Creating such astrophysical plasmas in laboratory settings for studying their nonlinear physical phenomena is challenging. However, their properties can be explored through numerical simulations that incorporate suitable plasma assumptions, reflecting observations in space and astrophysical environments.

Since plasma has fluid-like behavior. Thus, the continuity and momentum equations will be formulated via the mass andmomentum conservation laws. Also, the charged particles are interconnected to the electric field ($\vec{E} = -\nabla\phi$), where ϕ is the electrostatic potential . As a result, the continuity and the momentum equations are supplemented with the following Maxwell's equation $\vec{\nabla} \cdot \vec{E} = -4\pi\rho$, where ρ is the overall charge density on the surface. Already many researchers [75–81] have studied the propagation of ion acoustic soliton (IAS) by assuming various types of unmagnetized collisionless relativistic plasma environments. But, they ignored what happens with the propagation of IAS not only around the critical points but also around the critical composition of any specific parameters. Accordingly, finding the basic features of the nonlinear propagation of IAS by formulating the evolution equations of higher-order nonlinearity is still now in initial stage. Hence, this research work will focus on the basic features of nonlinear propagation of IAS not only around the critical points but also around the critical composition of any specific parameters in the collisionless plasmas under suitable plasma assumptions.

The main objective of the proposed research work is as follows:

i) To develop or considered the previously proposed models under some plasma assumptions in the unmagnetized collisionless multi-component plasmas.

ii) To implement the mathematical techniques for deriving the nonlinear evolution equations from the considered models.

iii) To investigate the nonlinear propagation of ion acoustic wave phenomena by the analytical solutions of nonlinear evolution equations.

1.11 OUTLINES OF THE THESIS

The thesis is categorized into the following six chapters:

Chapter 1 introduce fundamental concepts and the presence of three-component unmagnetized relativistic plasma in astrophysical, space, and laboratory settings. The problem of statements and velocity distribution functions are also discussed.

Chapter 2 describes the theoretical description of plasma phenomena, concept of fluid description, reductive perturbation method and soliton formation.

In Chapter 3, the relativistic plasma environment is proposed by the mixture of relativistic ion fluids and nonthermal distributed electrons as well as positrons to study the nonlinear IAS propagation with the consideration of Lorentz relativistic factor more than three terms described by KdVE.

Chapter 4 deals with the nonlinear IAS propagation around the critical values in relativistic plasmas described by mKdVE.

Chapter 5 examines the nonlinear IAS propagation around the super-critical values in relativistic plasmas described by KdVE involving quartic nonlinearity.

Finally, the concluding remarks and future direction are presented in Chapter 6.

Chapter 2: FLUID MODEL EQUATIONS AND METHODOLOGY

2.1 THEORETICAL DESCRIPTION OF PLASMA PHENOMENA

In the theoretical study of plasma phenomena, there are four main approaches, each offering different approximations tailored to specific conditions. One notable method, particle orbit theory, focuses on analyzing the movement of individual charged particles within defined electric and magnetic fields. While not strictly plasma theory, it provides valuable insights into the dynamics of charged particles under external influences. This approach has proven particularly effective in predicting behavior in extremely low-density plasmas, such as those found in the Van Allen radiation belts, the solar corona, cosmic rays, high-energy accelerators, and cathode ray tubes.

Given that plasmas consist of countless interacting particles, a statistical approach is crucial for developing a macroscopic description of plasma phenomena. This involves characterizing the system using a distribution function for the particles. Kinetic theory plays a significant role here, where the evolution of the distribution function in phase space is governed by kinetic equations like the Vlasov equation. This framework considers interactions among particles through smeared-out internal electromagnetic fields and typically neglects short-range correlations (close collisions).

In scenarios where plasma particles experience frequent collisions, leading each species to maintain a local equilibrium distribution function, a fluid-like description becomes applicable. This approach, known as two-fluid theory (or many-fluid theory for more species), treats the plasma as a mixture of interpenetrating fluids. It employs hydrodynamic equations, in addition to electrodynamic equations, to express conservation laws for mass, momentum, and energy for each particle species locally.

An alternative approach treats the entire plasma as a single conducting fluid, utilizing lumped macroscopic variables and corresponding hydrodynamic conservation equations. This framework is referred to as one-fluid theory and finds simplified application in studying very low-frequency phenomena in highly conductive fluids under magnetic fields, known as the magnetohydrodynamic approximation.

2.2 FLUID DESCRIPTION OF PLASMA

Describing plasmas using a single-particle approach becomes exceedingly complex. Essentially, a more statistical method is needed because it's impractical to track each particle individually. Fortunately, in many cases, this level of detail isn't necessary. Surprisingly, most observed plasma phenomena in real experiments can be adequately explained using a simplified fluid model. In this model, individual particle identities are ignored, and only the collective motion of fluid elements is considered. It's important to note that in plasmas, these fluid elements carry electrical charges, which distinguishes plasma fluid dynamics from traditional fluid dynamics.

2.3 BASIC OF FLUID DESCRIPTION

There is a fundamental difference between hydrodynamics and plasma fluid models. In hydrodynamics, molecules within a liquid are strongly coupled and undergo frequent collisions with their neighbors. This strong coupling allows fluid elements, comprising many molecules moving together, to follow streamlines defined by the flow pattern. Consequently, the diffusion of molecules within these fluid elements tends to occur relatively slowly.

In contrast, in an ideal plasma, electrons and ions do not experience frequent collisions with their nearest neighbors due to the rarity of Coulomb collisions. Instead, they predominantly respond to the average electric and magnetic fields generated by the collective behavior of many particles. This allows us to partition the plasma into small cells, but unlike in hydrodynamics, particles do not remain within their cells for extended periods. Electrons and ions typically exit a cell of size l after a transit time $T_t \equiv l/v_{th}$, where v_{th} is the thermal velocity, while particles from neighboring cells enter this volume. These cells serve as a metaphorical "bank account" where we track gains and losses of total particles, momentum, or heat content.

This approach offers a hydrodynamic-like description of plasma dynamics, although its analogy to real liquids has certain limitations. Depending on the circumstances, one can adopt a description where cells are fixed in a stationary frame of reference, or alternatively, transform to a moving frame of reference that tracks the mean flow velocity of the plasma. This flexibility enables the modeling approach to be adapted to various plasma phenomena and conditions.

2.3.1 CONTINUITY EQUATION

If the average velocity and number density of h species of a plasma system are v_h and n_h, respectively, then the total mass density of the plasma fluid [2,7] is defined as

$$\rho = n_h m_h, \tag{2.1}$$

and the average velocity of h species The continuity equation, rooted in the principle of mass conservation, states that the total number of particles (N) within a defined volume (V) remains constant unless there is a net flux of particles perpendicular to the surface (S) that encloses the volume. According to this principle, the equation is derived as:

$$\frac{\partial n_h}{\partial t} + \nabla.(n_h \mathbf{v}_h) = 0, \tag{2.2}$$

In this equation, (n_h) represents the number density of particles, and $(\mathbf{v}_h)$ denotes their thermal velocity. This equation is widely recognized in plasma physics as the continuity equation. It's important to note that the first term in Eq. (2.2) describes how the concentration of particles changes over time within the volume, while the second term indicates the divergence of the particle flux out of the volume.

2.3.2 MOMENTUM CONSERVATION EQUATION

In plasmas, the behavior of particles is influenced by several forces, with three main forces playing a significant role: the Lorentz force, the pressure gradient force, and collision effects. The Lorentz force governs how charged particles respond to electric and magnetic fields. It is given by:

$$\vec{F}_L = q\left(\vec{E} + \vec{v} \times \vec{B}\right), \tag{2.3}$$

In gas, the pressure P can be expressed as $P = nk_BT$, where n, k_B and T are the density, Boltzmann constant and temperature. This represents the force per unit area arising from the thermal motions of particles in the gas. The surrounding fluid exerts this force on an element within the gas. The pressure gradient force is given as $\vec{F}_p = -\nabla p/n$ by the gradient of pressure. The pressure gradient force arises from spatial variations in plasma pressure. It acts to accelerate particles from regions of higher pressure to regions of lower pressure, contributing to plasma flow and dynamics. Collision effects, on the other hand, involve interactions between particles due to collisions. These collisions can transfer momentum, heat, and can lead to plasma instabilities and changes in plasma properties over time. Hence one can immediately generalize the momentum equation for the plasma species in collisionless plasmas as

$$mn\left\{\frac{\partial \vec{v}}{\partial t} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}\right\} = qn\left(\vec{E} + \vec{v}\times\vec{B}\right) - \vec{\nabla}p. \tag{2.4}$$

For the relativistic collisionless unmagnetized plasmas, the above equation can be written as

$$mn\left\{\frac{\partial \vec{\gamma v}}{\partial t} + \left(\vec{v}\cdot\vec{\nabla}\right)(\gamma\vec{v})\right\} = qn\vec{E} - \vec{\nabla}p, \tag{2.5}$$

where $\gamma = \left(1 - \vec{v}^2/c^2\right)^{-1/2}$ and c is the speed of light.

2.3.3 POISSON'S EQUATION

The electric field $\vec{E}$ can be written in terms of electric potential ψ

$$\vec{E} = -\vec{\nabla}\psi. \tag{2.6}$$

Maxwell's equations, however, constitute a fundamental set of equations that delineate the characteristics of electric and magnetic fields. They form the foundation of classical electromagnetism and are essential in understanding various phenomena such as electromagnetic waves, radiation, and the interaction of fields with matter. By combing Gauss's Law for Electricity equation $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$ and Eq. (2.6), following equation is obtained

$$\nabla^2 \psi = -\rho/\varepsilon_0 \tag{2.7}$$

Eq. (2.7) is known as the Poisson's equation. ρ is the overall charge densities of the particles and ε_0 denotes the absolute permittivity of vacuum.

2.4 REDUCTIVE PERTURBATION TECHNIQUE

The reductive perturbation technique (RPT) involves expanding dependent quantities (such as densities, velocities, potentials, etc.) around their equilibrium positions using a small parameter ε. This parameter determines the strength of the perturbation and is crucial for balancing nonlinear and dispersive effects. The expansion is performed in terms of powers of ε in stretched coordinates, aligning with the transformation to the wave frame of reference. Higher powers of ε indicate slower variation of physical quantities compared to those with lower powers. Many physical systems are described by complex equations of motion involving multiple dependent variables. To simplify such systems, Washimi and Taniuti proposed a systematic procedure [20] to reduce general nonlinear evolution equations into more manageable forms. This reduction assumes small wave amplitudes, allowing the original hyperbolic system to be transformed into simpler nonlinear equations like the KdVE, Burgers equation, or the nonlinear Schrödinger equation. The RPT relies on practical experience rather than strict mathematical rules for choosing relevant scales. Normalization of all variables in the problem to make them dimensionless is advisable before applying the reductive perturbation method. This simplifies the analysis by reducing the number of constants that need to be considered during calculations. In the next step, one can expand all the perturb quantities in terms of ε. Some expansion of perturb quantities, like density n, velocity v and electrostatic potential ϕ are given below.

$$\begin{aligned} n &= n_0 + \varepsilon n_1 + \varepsilon^2 n_2 + \varepsilon^3 n_3 + ... \\ v &= \varepsilon v_1 + \varepsilon^2 v_2 + ... \qquad . \\ \phi &= \varepsilon \phi_1 + \varepsilon^2 \phi_2 + ... \end{aligned} \tag{2.8}$$

In the context of weakly dispersive waves and the theory of solitons, in the first approximation, the wave moves with a phase velocity that is independent of wave number k. This velocity remains constant in a frame of reference moving with the wave. Within this frame, the wave's evolution is primarily governed by weak dispersion and weak nonlinearity. Over longer timescales, particularly in the case of solitons, there is a delicate balance between dispersion and nonlinearity. This balance ensures that the wave retains its shape and travels without significant distortion over considerable distances. This physical concept is effectively captured by transforming to coordinates [40]

$$\xi = \varepsilon^r (x - \lambda t); \tau = \varepsilon^{3r} t. \tag{2.9}$$

where λ is the phase velocity and r is any real number. With the suitable choice of r and implement the expansion of physical variables to the model equations, one can derive various types of physical equations to study the soliton propagation in the plasmas.

2.5 FORMATION OF SOLITON

Solitons arise from the delicate equilibrium between a medium's nonlinearity and dispersion. Typically, in dispersive media, linear waves propagate over long distances, but dispersion causes them to spread due to the phase velocity becoming dependent on wave number k. In nonlinear media, as wave amplitudes increase, nonlinear effects become significant. These effects primarily steepen the leading edge of waves, often resulting in wave breaking. Figure 2.1a illustrates how dispersion spreads an initial waveform, while Figure 2.1b shows nonlinear effects causing waveform steepening. Figure 2.1c demonstrates how the interplay of dispersion and nonlinearity leads to the formation of solitons, stable wave structures.

Plasma generally behaves nonlinearly, and although most plasma waves exhibit dispersion, not all show soliton solutions. Ion acoustic waves in plasma, specifically ion acoustic solitons, exhibit soliton behavior and serve as a typical example. These solitons are commonly described by the KdVE, originally formulated by D. J. Korteweg and G. de Vries in 1895 [82].

2.5.1 KDVE

The KdVE represents as a pivotal nonlinear partial differential equation with wide-ranging applicability across various physical systems. Its extensive examination and application span different fields including mathematics, physics, and engineering. In particular, within plasma dynamics, the KdVE is as useful in explaining the behavior of nonlinear waves in several plasma environments. Researchers have broadened the scope of the KdVE by adding phenomena exceeding its traditional formulation. In addition, the RPT has been utilised for deriving the KdVE and its adapted iterations across varied plasma spaces, thereby enriching knowledge of wave dynamics within these types of conditions.

Various studies have explored different transformations and solutions related to the KdVE, including the Sharma-Tasso-Olver equation and higher order KdVE, showcasing the versatility and applicability of the KdV framework in diverse contexts [83–85]. The KdVE has been generalized to include mixed dispersion and shoaling terms, broadening its range of applications to describe phenomena in deep waters [86]. Soliton solutions and peakon solutions for KdV-like equations have been investigated, demonstrating the rich mathematical structure and solvable properties of equations related to the KdVE [87]. The KdVE serves as a cornerstone in the study of nonlinear wave phenomena, with its extensions and applications contributing to a deeper understanding of wave dynamics in various physical systems.

Nonlinearity and dispersion are two of the most important properties of plasma. The nonlinear dispersion equation developed by Korteweg and de Vries in various physical situations, including problems relevant to plasma physics, can be written in its simplest

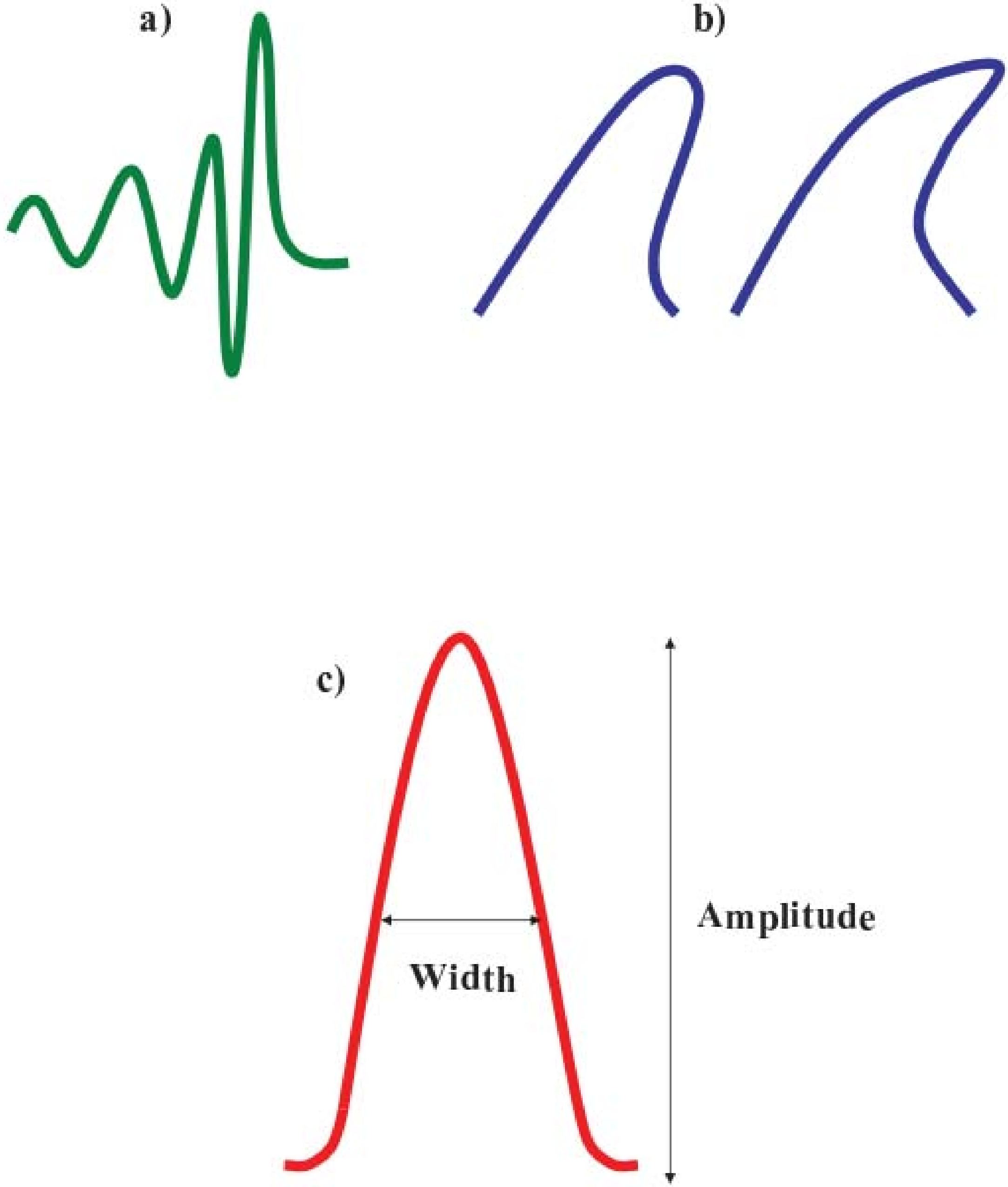

Figure 2.1: a) Linear dispersive wave: when the phase velocity becomes k dependent, the wave disperses, b) Nonlinear wave: with the increase in amplitude the wave steepens and has a tendency to break and c) Soliton..

form [88],

$$\frac{\partial \phi}{\partial \tau} + P\phi\frac{\partial \phi}{\partial \xi} + Q\frac{\partial^3 \phi}{\partial \xi^3} = 0 \tag{2.10}$$

Here, ξ and τ represent independent variables, and P and Q stands for real, nonzero constants. Equation (2.10) reveals the presence of nonlinearity via the term $\phi\frac{\partial \phi}{\partial \xi}$, while dispersion is depicted by $\frac{\partial^3 \phi}{\partial \xi^3}$.

Historically, the KdVE was initially introduced by Boussinesq in 1877 and later rediscovered by Korteweg and de Vries in 1895 [89], who identified the simplest one-soliton solution. Significant progress in understanding the equation and its solutions came from Zabusky and Kruskal's [90] computer simulations in 1965 and the development of the inverse scattering transform in 1967. The KdVE can be solved using the inverse scattering method [91], which was developed by Gardner, Greene, Kruskal, and Miura [92]. Kruskal and Zabusky [90, 92] derived this equation (2.24) for one dimensional acoustic waves in anharmonic crystals. Also, Washimi and Taniuti [93] showed that it describes weakly nonlinear one dimensional acoustic waves traveling near the ion sound speed in plasma physics. Given its various applications, generalizations of the KdVE are necessary. Rescaling the equation (2.10) with $\varepsilon \rightarrow \xi Q^{\frac{1}{3}}$ and $\phi \rightarrow \frac{\phi}{PQ^{-\frac{1}{3}}}$ to give coefficients of unity in front of each term, i.e.

$$\frac{\partial \phi}{\partial \tau} + \phi\frac{\partial \phi}{\partial \xi} + \frac{\partial^3 \phi}{\partial \xi^3} = 0 \tag{2.11}$$

The steady-state solution of the standard KdV equation (2.11) is typically obtained in the literature by transforming the variables ξ and τ into a new coordinate $\eta = (\xi - V_p t)$, where V_p is a constant. Integrating equation (2.11) one can obtain

$$\phi(\eta) = 3V sech^2\left[\frac{(\xi - V_p t)}{2}\right] \tag{2.12}$$

In equation (2.12), the amplitude, width, and speed of the solitary wave are proportional to $V_p^{-\frac{1}{2}}$, and V_p, respectively. Computer simulations [90, 94, 95] of equation (2.12) reveal that solitary wave solutions (solitons) given by equation (2.26) are essential for the system's time evolution, interacting nonlinearly and maintaining their identity. The KdVE serves as a cornerstone in the study of nonlinear wave phenomena, with its extensions and applications contributing to a deeper understanding of wave dynamics in various physical systems.

2.5.2 MKDVE

A mKdVE, characterized by a cubic nonlinearity term instead of a quadratic one, can be derived by considering the next higher order of ε along with the lower orders used in the derivation of the KdVE. However, different stretched coordinates, $\xi = \varepsilon(x - V_p t)$ and $\tau = \varepsilon^3 t$, are used to derive the mKdV equation. This change does not affect the linear order, which remains the same, but introduces the cubic term in the next order. Following a similar elimination process, we obtain the mKdVE as follows:

$$\frac{\delta\phi}{\delta\tau} + A\phi^2\frac{\delta\phi}{\delta\xi} + B\frac{\delta^3\phi}{\delta\xi^3} = 0. \tag{2.13}$$

The solution of the above mKdV equation (2.13) can be written as

$$\phi = \sqrt{\frac{6V_p}{A}} sech\left\{\frac{\eta}{W}\right\}$$

Here V_p is the velocity of nonlinear structure, A is nonlinearity coefficient, B is dispersion coefficient and the width W is given as $W = \frac{V_p}{B}$.

Chapter 3: ION ACOUSTIC SOLITON IN AN UNMAGNETIZED RELATIVISTIC PLASMA

3.1 INTRODUCTION

In recent decades, the study of wave propagation in multi-component plasmas has become a significant focus in plasma physics research. Extensive investigations [96–103] have explored the characteristics of nonlinear waves using various plasma models. Nonlinear phenomena are crucial in forming distinct coherent structures. These structures are of both theoretical and experimental interest. For instance, in laser-plasma interactions, nonlinear electrostatic [101] or electromagnetic waves [102], collisionless solitons [104], and phase space holes involving ions and electrons [103] have been observed. Solitary waves, which maintain their shape and size over time in a frame moving at the group velocity of the wave, are primarily sustained by a balance between nonlinearity and dispersion, with minimal dissipation. Solitons, a type of nonlinear solitary wave, retain their structure even after interacting with another soliton, owing to the delicate balance between nonlinearity and dispersion. They offer insights into wave-plasma interactions and have been extensively studied both theoretically and experimentally for decades, albeit with limited research in relativistic plasmas.

Relativistic effects significantly alter soliton behavior in plasmas where electron or ion velocities approach the speed of light. Such plasmas are encountered in scenarios like high power laser-plasma interactions and in space environments, including Earth's magnetosphere and the solar atmosphere. Despite the prevalence of plasmas with different temperatures and masses in space, research on IASs and other nonlinear structures remains sparse [105]. Theoretical frameworks often utilize variations of equations like the Korteweg-de Vries equation (KdVE), modified KdV (mKdVE), or nonlinear Schrödinger equations to study solitary wave behavior in plasmas. Experimental observations complement these theoretical studies in diverse plasma conditions, including multi-species and dusty plasmas in space. Given the varied plasma conditions and relativistic velocities in astrophysical and space environments, there is a pressing need to investigate IASs in these systems. However, fundamental aspects of IA waves in relativistic plasmas, along with their nonlinear evolution equations, have already been explored in Refs. [33,106–118]. Previous studies [33,106–118] have extensively investigated the propagation characteristics of IASs in UMRPs by expanding the LRF up to two or three terms. However, to minimize truncation errors, it is necessary to consider expanding the LRF beyond three terms. This is crucial because many astrophysical plasmas occur on large spatial scales. Therefore, further research is warranted to explore the previously proposed theoretical model equations for UMRPs.

Thus, this chapter addresses the influences of the plasma parameters on the propagation characteristics of nonlinear IASs by deriving KDVE from the considered plasma environments along with the consideration of expansion of LRF more than three terms.

3.2 THEORETICAL MODEL EQUATIONS

A collisionless UMRP, consisting of inertialess nonthermal electrons as well as positrons and inertial relativistic warm ions is considered. Due to the inertialess electrons and positrons, one can assume Cairns velocity distribution to determine the density functions of such charged particles. As a result, one can study the contribution of restoring force that provided the thermal pressure of electrons and positrons via the charge neutrality condition. Based on the Cairns velocity distribution, the following concentrations of electrons and positrons are defined in Refs. [38,119–121]:

$$\begin{aligned} N_e &= N_{e0}\left[1 - \frac{4\alpha_e}{1+3\alpha_e}\left(\frac{e\phi}{T_e}\right) + \frac{4\alpha_e}{1+3\alpha_e}\left(\frac{e\phi}{T_e}\right)^2\right]\exp\left(\frac{e\phi}{T_e}\right), \\ N_p &= N_{p0}\left[1 + \frac{4\alpha_p}{1+3\alpha_p}\left(\frac{e\phi}{T_p}\right) + \frac{4\alpha_p}{1+3\alpha_p}\left(\frac{e\phi}{T_p}\right)^2\right]\exp\left(-\frac{e\phi}{T_p}\right), \end{aligned} \tag{3.1}$$

where $N_e(N_p)$ stands for electrons (positrons) density, $N_{e0}(N_{p0})$ stands for the unperturbed electron (positron) concentration, $T_e(T_p)$ indicates the electron (positron) temperature, ϕ

denotes the electrostatic potential, e is the charge of electron and $\alpha_{e,p} > -1/3$ calculates the population of nonthermal electrons and positrons, respectively. However, one can derive the ion continuity and momentum conservation equations due to the inertia of relativistic ions. As a result, one can use the following normalized fluid equations to study the nonlinear dynamics of IAWs around the CVs of any specific plasma parameter:

$$\frac{\partial N_i}{\partial t} + \frac{\partial (N_i U_i)}{\partial x} = 0, \tag{3.2}$$

$$\frac{\partial(\gamma U_i)}{\partial t} + U_i \frac{\partial(\gamma U_i)}{\partial x} + \frac{3\delta_{ie}}{(1-p)^2} N_i \frac{\partial N_i}{\partial x} = -\frac{\partial \phi}{\partial x}, \tag{3.3}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \left(1 - \beta_e \phi + \beta_e \phi^2\right) e^{\phi} - p\left(1 + \beta_p \sigma \phi + \beta_p \sigma^2 \phi^2\right) e^{-\sigma\phi} - N_i, \tag{3.4}$$

where the phase velocity is assumed to have much lower values than the electron and positron thermal velocities but much higher values than the ion thermal velocity, $\beta_e = 4\alpha_e/(1+3\alpha_e)$ and $\beta_p = 4\alpha_p/(1+3\alpha_p)$. In the above equations, N_i, U_i and ϕ are the normalized ion number density, ion fluid velocity and electrostatic potential normalized by $N_i \to N_i/N_{e0}$, $U_i \to U_i/C_s$, and $\phi \to e\phi/k_B T_e$, where $C_s = \sqrt{(k_B T_e/m_i)}$, m_i, e, and k_B indicate the ion acoustic speed, ion mass, electronic charge and Boltzmann constant, respectively. The space (x) and time (t) involved in the above equations are normalized by $\lambda_{De} = \sqrt{k_B T_e/(4\pi N_{e0} e^2)}$ and $\omega_{pi}^{-1} = \sqrt{m_i/(4\pi N_{e0} e^2)}$, respectively. $\gamma = 1/\sqrt{1 - U_i^2/c^2}$ is the Lorentz relativistic factor (LRF). Additionally, the involving other parameters in the above equations are determined as $p = N_{p0}/N_{e0}$, $\delta_{ie} = T_i/T_e$ (T_i is the ion temperature) and $\sigma = T_e/T_p$.

3.3 FORMATION OF KDVE AND STATIONARY SOLITON SOLUTION

For deriving the KdVE by simplifying Eqs. (3.2) to (3.4), one can use the scaling of dependent unknown variables via the new stretched coordinates [110] as

$$\xi = \varepsilon^{\frac{1}{2}} \left(x - V_p t \right), \; \tau = \varepsilon^{\frac{3}{2}} t, 0 < \varepsilon < 1, \tag{3.5}$$

where V_p is the linear phase velocity of the perturbation mode normalized by C_s and ε measures the weakness of the dissipation. Using Eq. (3.5) into Eqs. (3.2) -(3.4), one can convert to the Eqs. (3.2) -(3.4) in the new forms involving the new stretched coordinates as

$$\varepsilon^{3/2} \frac{\partial N_i}{\partial \tau} - \sqrt{\varepsilon} V_p \frac{\partial N_i}{\partial \xi} + \sqrt{\varepsilon} \frac{\partial (N_i U_i)}{\partial \xi} = 0, \tag{3.6}$$

$$\varepsilon^{3/2} \frac{\partial \gamma U_i}{\partial \tau} - \sqrt{\varepsilon} \frac{\partial \gamma U_i}{\partial \xi} + U_i \sqrt{\varepsilon} \frac{\partial \gamma U_i}{\partial \xi} + \sqrt{\varepsilon} N_i \frac{3\delta_{ie}}{1-p} \frac{\partial N_i}{\partial \xi} = -\sqrt{\varepsilon} \frac{\partial \phi}{\partial \xi}, \tag{3.7}$$

$$\varepsilon \frac{\partial^2 \phi}{\partial \xi^2} = \left(1 - \beta_e \phi + \beta_e \phi^2\right) e^{\phi} - p \left(1 + \beta_p \sigma \phi + \beta_p \sigma^2 \phi^2\right) e^{-\sigma \phi} - N_i. \tag{3.8}$$

Then, one can use the following expansions of perturbed quantities N_i, U_i, and ϕ, which are involved in the power series of ε [110]:

$$N_i = (1-p) + \sum_{j=1}^{\infty} \varepsilon^j N_i^{(j)}, U_i = U_{i0} + \sum_{j=1}^{\infty} \varepsilon^j U_i^{(j)}, \; \phi = \sum_{j=1}^{\infty} \varepsilon^j \phi^{(j)}. \tag{3.9}$$

As a result, one can derive the different set of partial differential equations (PDEs) by taking several order of ε. The lowest order of ε PDE yields,

$$-\left(V_p - U_{i0}\right) \frac{\partial N_i^{(1)}}{\partial \xi} + (1-p) \frac{\partial U_i^{(1)}}{\partial \xi} = 0, \tag{3.10}$$

$$-\left(V_p - U_{i0}\right) \gamma_1 \frac{\partial U_i^{(1)}}{\partial \xi} + \frac{3\delta_{ie}}{1-p} \frac{\partial N_i^{(1)}}{\partial \xi} = -\frac{\partial \phi^{(1)}}{\partial \xi}, \tag{3.11}$$

$$-C_1\phi^{(1)}+N_i^{(1)}=0, \tag{3.12}$$

where

$$\gamma_1=\sum_{m=0}^{\infty}\frac{(-1)^m\left(-\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right)}{m!\left(-\frac{1}{2}-m\right)\Gamma\left(-\frac{1}{2}-m\right)}(2m+1)\,\beta_{ic}^{2r},$$

$$\beta_{ic}=\frac{U_{i0}}{c},$$

$$C_1=[(1-\beta_e)+p\sigma(1-\beta_p)].$$

From Eq. (3.10)-(3.12), the first order perturbed quantities for N_i and U_i can be defined as

$$N_i^{(1)}=\frac{(1-p)}{k}\phi^{(1)}=C_1\phi^{(1)},\ U_i^{(1)}=\frac{(V_p-U_{i0})}{k}\phi^{(1)}, \tag{3.13}$$

where $k=[(V_p-U_{i0})^2\gamma_1-3\delta_{ie}]$. Also, V_p can be expressed as

$$V_p=U_{i0}+\left\{\frac{3\delta_{ie}}{\gamma_1}+\frac{(1-p)}{\gamma_1[(1-\beta_e)+p\sigma(1-\beta_p)]}\right\}^{1/2}. \tag{3.14}$$

Again, the next order of ε PDEs are obtained as follows:

$$-(V_p-U_{i0})\frac{\partial N_i^{(2)}}{\partial\xi}+(1-p)\frac{\partial U_i^{(2)}}{\partial\xi}+\frac{\partial}{\partial\xi}\left(N_i^{(1)}U_i^{(1)}\right)+\frac{\partial N_i^{(1)}}{\partial\tau}=0, \tag{3.15}$$

$$\begin{aligned}&-\gamma_1(V_p-U_{i0})\frac{\partial U_i^{(2)}}{\partial\xi}+\frac{3\delta_{ie}}{(1-p)}\frac{\partial N_i^{(2)}}{\partial\xi}+\{\gamma_1-\gamma_2(V_p-U_{i0})\}U_i^{(1)}\frac{\partial U_i^{(1)}}{\partial\xi}\\&+\frac{3\delta_{ie}}{(1-p)^2}N_i^{(1)}\frac{\partial N_i^{(1)}}{\partial\xi}+\gamma_1\frac{\partial U_i^{(1)}}{\partial\tau}=-\frac{\partial\phi^{(2)}}{\partial\xi},\end{aligned} \tag{3.16}$$

$$\frac{\partial^2\phi^{(1)}}{\partial\xi^2}=C_1\phi^{(2)}+C_2\left\{\phi^{(1)}\right\}^2-N_i^{(2)}, \tag{3.17}$$

where $C_2 = \left(1 - p\sigma^2\right)/2$ and

$$\gamma_2 = \sum_{m=1}^{\infty} \frac{(2m+1)!}{(2m-1)!} \frac{(-1)^m \left(-\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right)}{m!\left(-\frac{1}{2}-m\right)\Gamma\left(-\frac{1}{2}-m\right)} \frac{\beta_{ic}^{2m}}{U_{i0}}.$$

Now, combining Eqs. (3.15) -(3.17) with the help of Eqs. (3.13), the following KdVE is obtained:

$$\frac{\partial \phi^{(1)}}{\partial \tau} + P\phi^{(1)} \frac{\partial \phi^{(1)}}{\partial \xi} + Q \frac{\partial^3 \phi^{(1)}}{\partial \xi^3} = 0, \tag{3.18}$$

where

$$P = \frac{1}{(V_P - U_{i0})\,k} \left[\frac{3\delta_{ie}}{2\gamma_1} - \frac{k^3 C_2}{\gamma_1 (1-p)} + \frac{\{3\gamma_1 - \gamma_2 (V_P - U_{i0})\}(V_P - U_{i0})^2}{2\gamma_1} \right]$$

$$Q = \frac{k^2}{2\gamma_1 (V_p - U_{i0})(1-p)}$$

The well-established soliton solution of Eq. (3.18) is obtained by considering a reference frame $\zeta = \xi - U_0 \tau$ (U_0 stands for the constant reference speed) and $\phi^{(1)} \to 0$, $\frac{d\phi^{(1)}}{d\zeta} \to 0$, …..as $\zeta \to \pm\infty$ in the following form:

$$\phi^{(1)} = \phi_l sech^2 \left\{ \frac{\zeta}{\phi_m} \right\}, \tag{3.19}$$

where $\phi_l = (3U_0/P)$ and $\phi_m = \sqrt{(Q/U_0)}$ are the amplitudes and widths of the soliton, respectively.

3.4 RESULTS AND DISCUSSIONS

The investigation into electrostatic IASs involves deriving the KdVE to understand their nature. It is intriguing and crucial to identify the exact soliton wave solution for KdVE. Since KdVE is integrable, its solution can be determined through direct integration. Therefore, the established solution of KdVE is considered. The impact of parameters such as $p(= N_{p0}/N_{e0})$, $\delta_{ie}(= T_i/T_e)$, $\sigma(= T_e/T_p)$, $\beta = \beta_e = \beta_p$ (population of nonthernal electrons and positrons) and $\beta_{ic}(= U_{i0}/c)$, along with the expansion of LRF more than three terms, on the propagation of electrostatic IASs in plasma is described using the analytical solution of KdVE. The influence of these parameters on the amplitude and width of IASs is illustrated in figures 3.1-3.6, with constant values chosen for other plasma parameters. In this analysis, numerical values of plasma parameters are selected based on relevant references pertaining to astrophysical and space environments. Figures 3.1, 3.2, 3.3 and 3.4 illustrate the influence of p, δ_{ie}, σ and β on the 3D propagation of nonlinear electrostatic solitons in the considered relativistic plasma, while keeping the remaining parameter constant. In contrast, Figures 3.5, 3.6, 3.7 and 3.8 demonstrate the impact of p, δ_{ie}, σ and β on the normalized electric field $\mathbf{E} = -\nabla\phi^{(1)}$ with the choice of the remaining parameter constant. Finally, Figures 3.9 and 3.10 illustrate 2D shaped of electrostatic soliton propagation and the normalized electric field with the variation of time.

From Figures 3.1 to 3.4, it is evident that the plasma sustains propagation of both compressive and rarefactive solitons. The amplitude and width of electrostatic IASs decrease with increasing density and temperature ratios, as well as with increasing relativistic streaming factors. In the physical point of view, solitons are nonlinear waves that can be compressive (where density increases) or rarefactive (where density decreases). The observation that the plasma supports both types of solitons indicates the complex interplay between different physical parameters influencing wave propagation. Additionally, the amplitude refers to the maximum disturbance caused by the soliton, while the width corresponds to the spatial extent over which this disturbance occurs. In the context of electrostatic IASs, which are governed by the balance between nonlinear and dispersive effects in plasma, the amplitude and width of these solitons are crucial indicators of their stability and behavior. Increasing density and temperature ratios generally lead to stronger interactions within the plasma, affecting the propagation characteristics of solitons. Specifically, as these ratios increase, the amplitude of IASs tends to decrease. This could be due to enhanced plasma screening effects or changes in the ion acoustic speed, influencing how the solitons maintain their structure and energy. Moreover, when particles in the plasma exhibit significant relativistic velocities (relativistic streaming), it alters the plasma dynamics and can affect the propagation of solitons. Increasing relativistic streaming factors typically lead to a increase in soliton amplitude and width, indicating a more complex and potentially less stable wave behavior in the plasma.

From Figures 3.5 to 3.8, it is evident that the plasma sustains the semi-kink shaped propagation of the normalized electric field. the amplitude and width of he normalized electric field decrease with increasing density and temperature ratios. It is observed from Figures 3.9 and 3.10 that both electrostatic potential and normalized electric field are behaves pulse like shaped with the increase of time. It is also clearly found that the electrostatic potential and the normalized electric field are possessed the pulse-like structures, as it is expected. It is noted that the expansion of LRF are considered up to 20 terms in this presented studies. Whereas, the expansion of LRF has been considered up to either two or three terms in this previous studies. Thus, the work has been made in this chapter would be helpful to understand the nonlinear propagation features of IASs in the presence of relativistic ion fluids and nonthermal electrons as well as positrons in many astrophysical and space environments, because many astrophysical plasmas occur on large spatial scales.

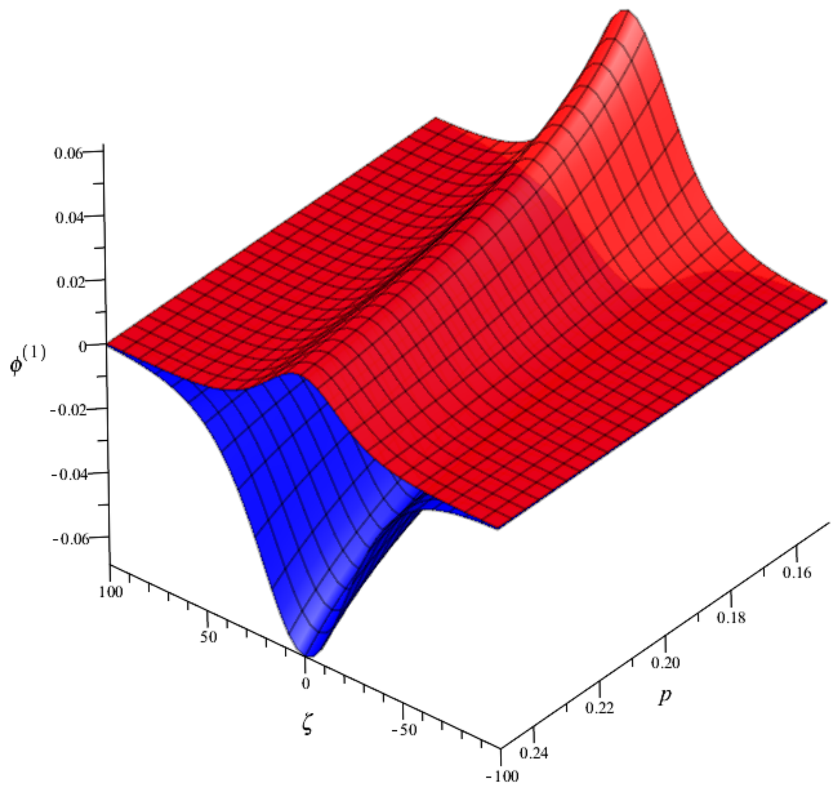

Figure 3.1: Electrostatic potential with regards to p and ζ with $\beta = 0.7$, $\delta_{ie} = 0.5$, $\sigma = 1.5$, $U_0 = 0.0075$, $U_{i0} = 0.3 \times 10^8$ and $\beta_{ic} = 0.1$. And $\beta = 0.7$, $\delta_{ie} = 0.01$, $\sigma = 1$, $U_0 = 0.0075$, $U_{i0} = 0.3 \times 10^8$ and $\beta_{ic} = 0.1$.

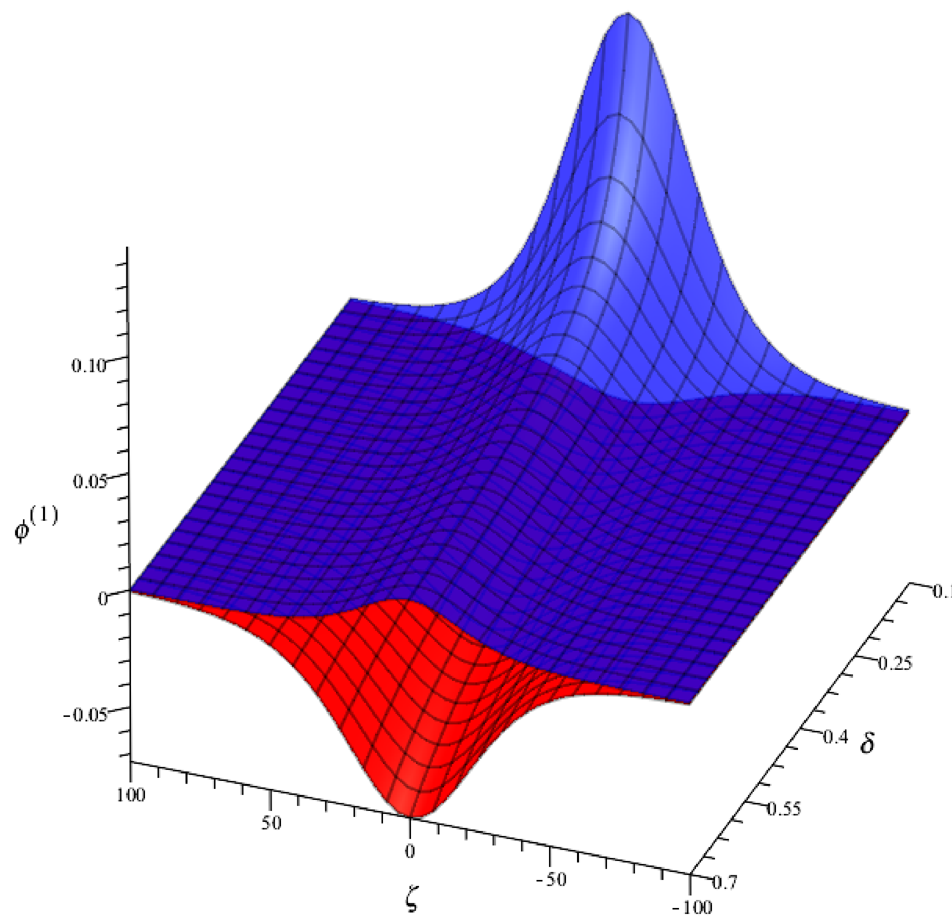

Figure 3.2: The normalized electric field with regards to δ_{ie} and ζ with $\beta = 0.7$, $p = 0.3$, $\sigma = 1$, $U_0 = 0.0075$, $U_{i0} = 0.3 \times 10^8$ and $\beta_{ic} = 0.1$. And $\beta = 0.7$, $p = 0.1$, $\sigma = 1$, $U_0 = 0.0075$, $U_{i0} = 0.3 \times 10^8$ and $\beta_{ic} = 0.1$.

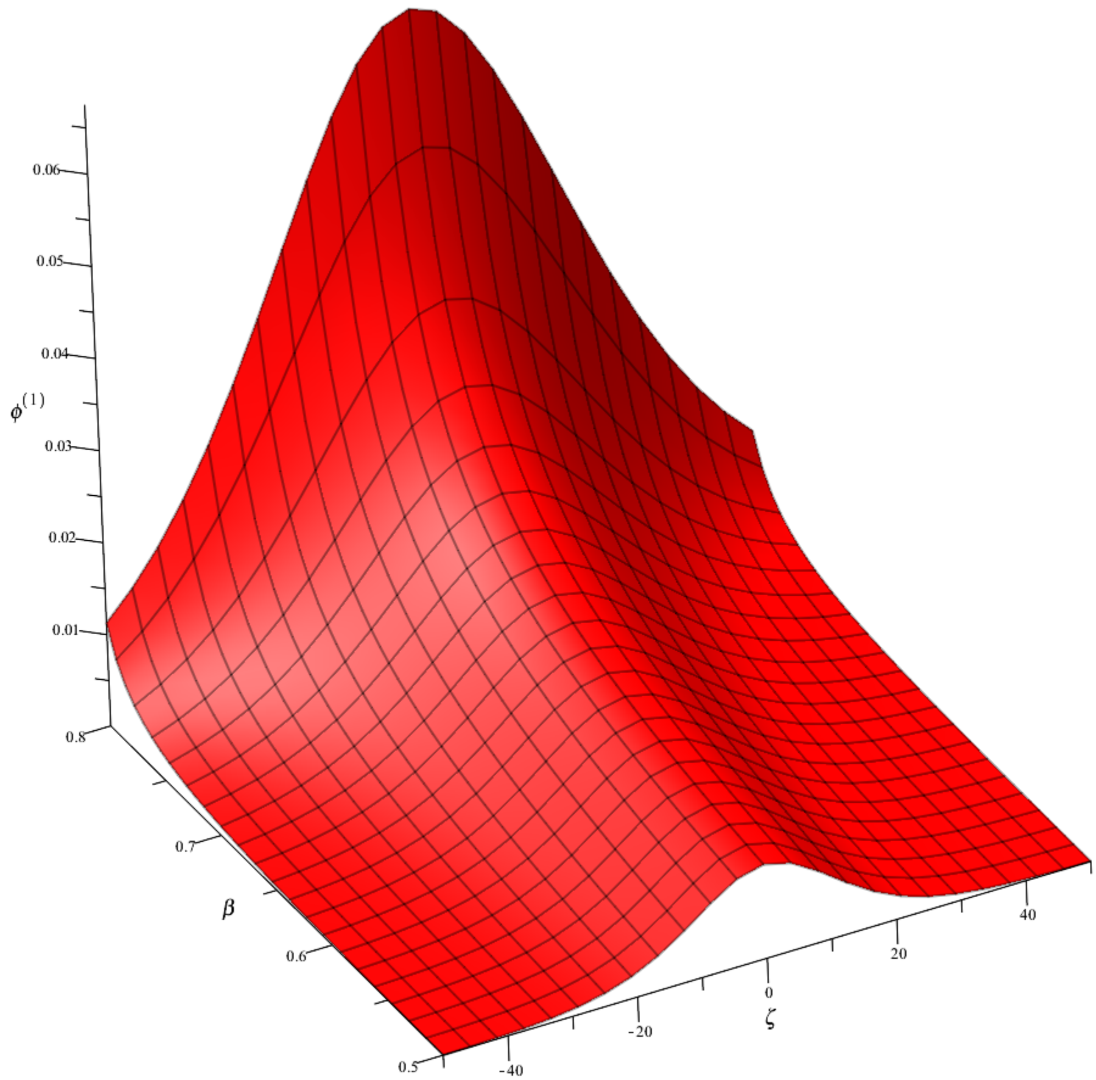

Figure 3.3: The normalized electric field with regards to $\beta_e = \beta_p = \beta$ and ζ with $p = 0.5$, $\delta_{ie} = 0.1$, $\sigma = 1$, $U_0 = 0.0075$, $U_{i0} = 0.3 \times 10^8$ and $\beta_{ic} = 0.1$.

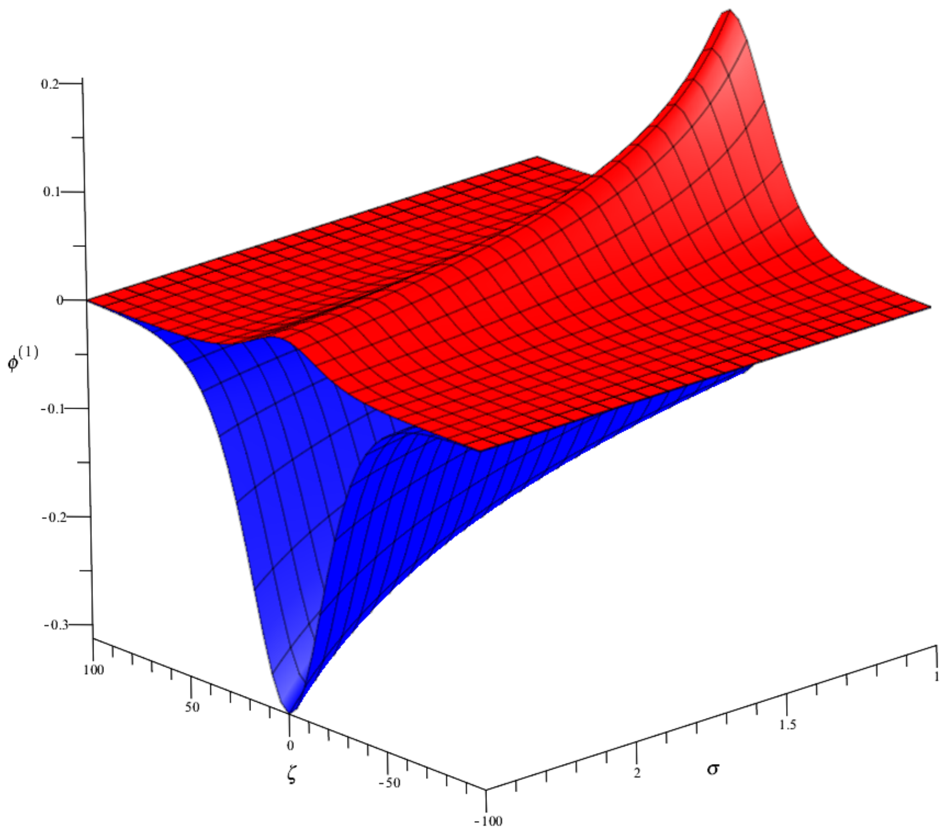

Figure 3.4: The normalized electric field with regards to σ and ζ with $p = 0.07$ and 0.01, $\delta_{ie} = 0.05$, $\beta = 0.5$, $U_0 = 0.0075$, $U_{i0} = 0.9 \times 10^8$ and $\beta_{ic} = 0.3$.

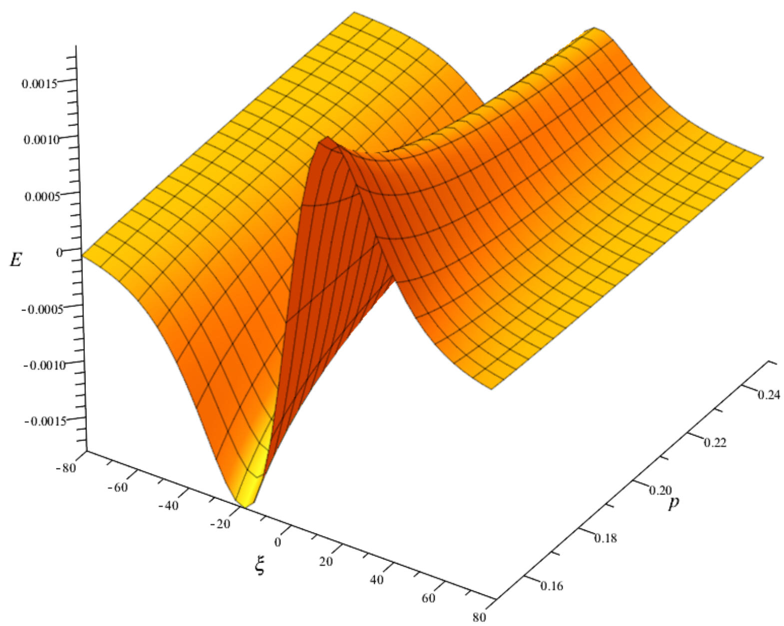

Figure 3.5: The normalized electric field with regards to p and ζ with $\beta = 0.7$, $\delta_{ie} = 0.5$, $\sigma = 1.5$, $U_0 = 0.0075$, $U_{i0} = 0.9 \times 10^8$ and $\beta_{ic} = 0.3$.

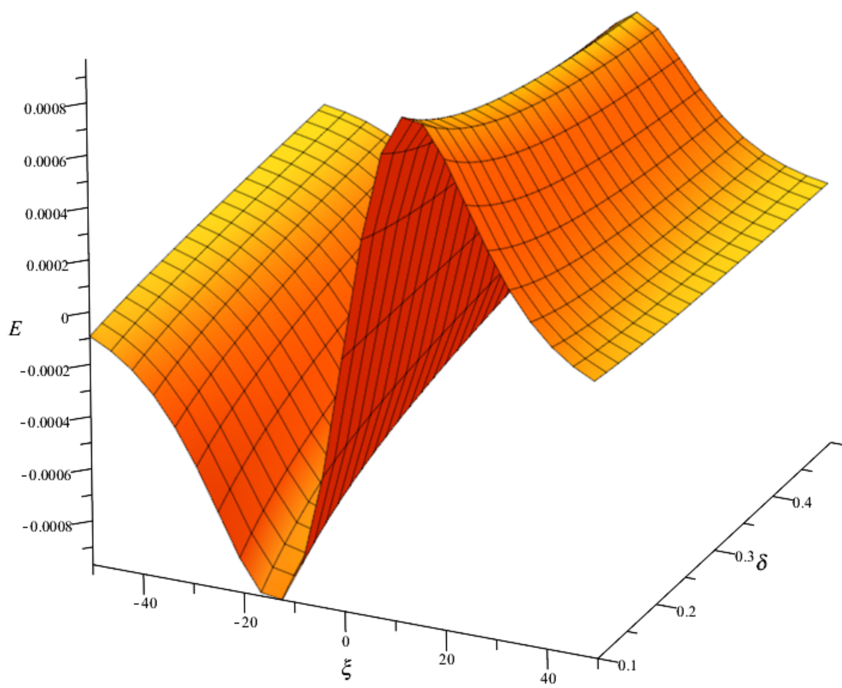

Figure 3.6: The normalized electric field with regards to δ_{ie} and ζ with $\beta = 0.5$, $p = 0.2$, $\sigma = 1$, $U_0 = 0.0075$, $U_{i0} = 0.3 \times 10^8$ and $\beta_{ic} = 0.1$.

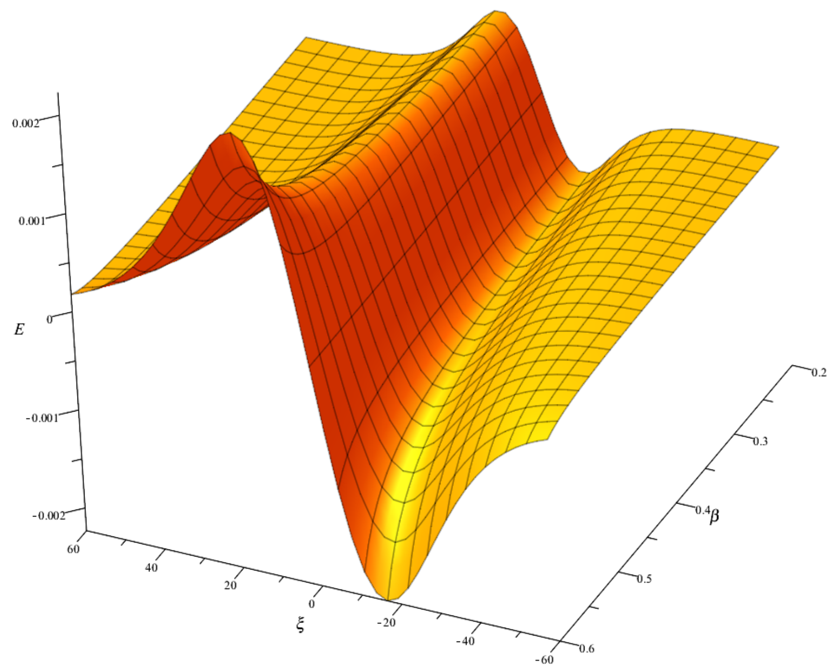

Figure 3.7: The normalized electric field with regards to $\beta_e = \beta_p = \beta$ and ζ with $p = 0.2$, $\delta_{ie} = 0.1$, $\sigma = 1$, $U_0 = 0.0075$, $U_{i0} = 0.9 \times 10^8$ and $\beta_{ic} = 0.3$.

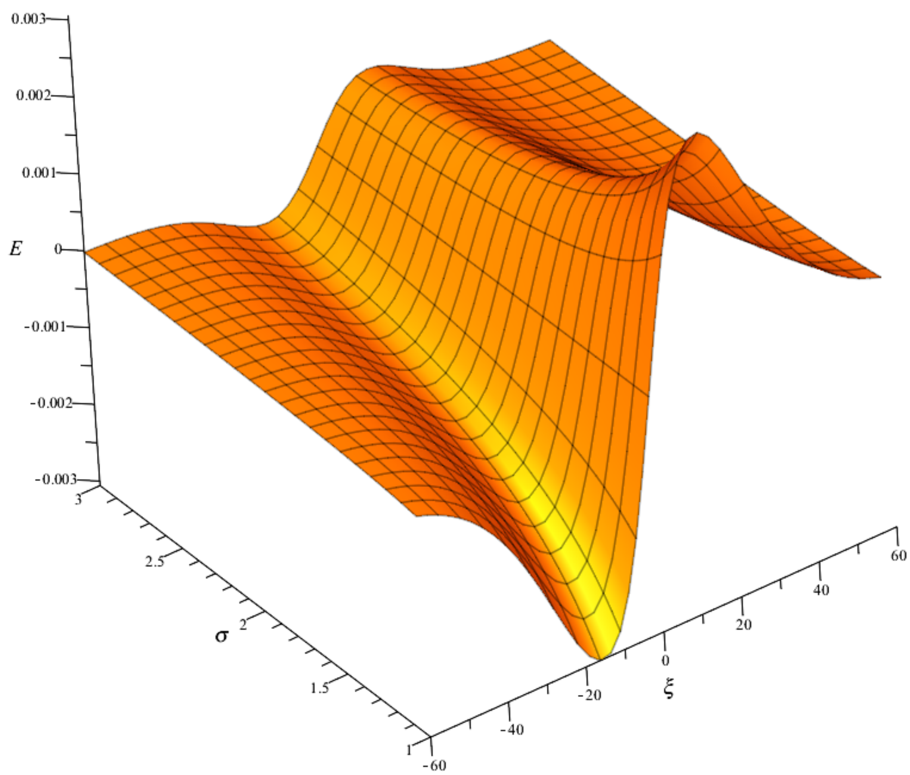

Figure 3.8: The normalized electric field with regards to σ and ζ with $p = 0.1$, $\delta_{ie} = 0.05$, $\beta = 0.5$, $U_0 = 0.0075$, $U_{i0} = 0.9 \times 10^8$ and $\beta_{ic} = 0.3$.

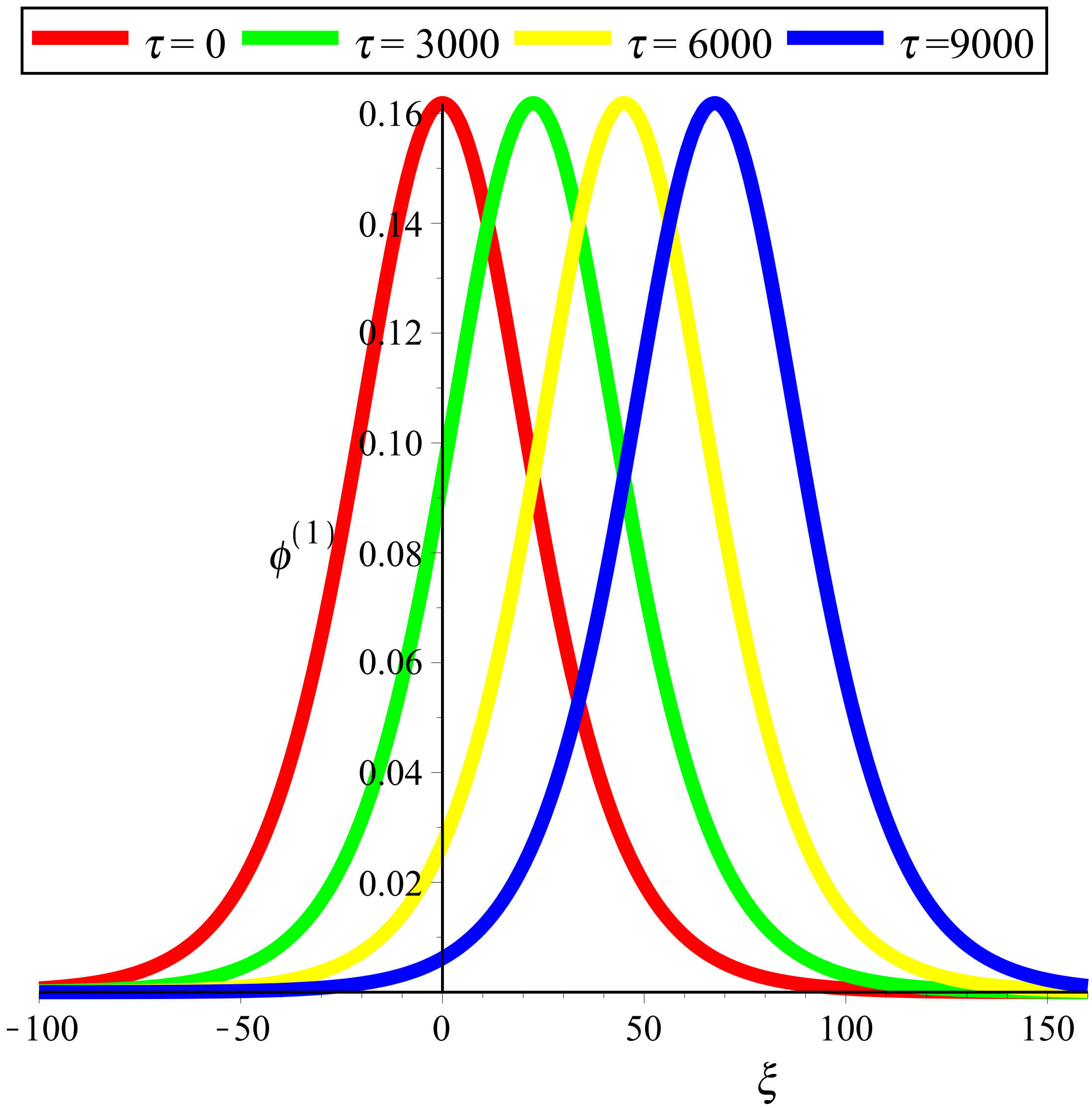

Figure 3.9: Variation of electrostatic potential with regards to time. The parametric values of the parameters are considered as $\beta_{ic} = 0.1$, $\delta_{ie} = 0.1$, $\sigma = 1$, $U_0 = 0.0075$, $U_{i0} = 0.3 \times 10^8$ and $\beta_e = \beta_p = 0.7$ and $p = 0.3$.

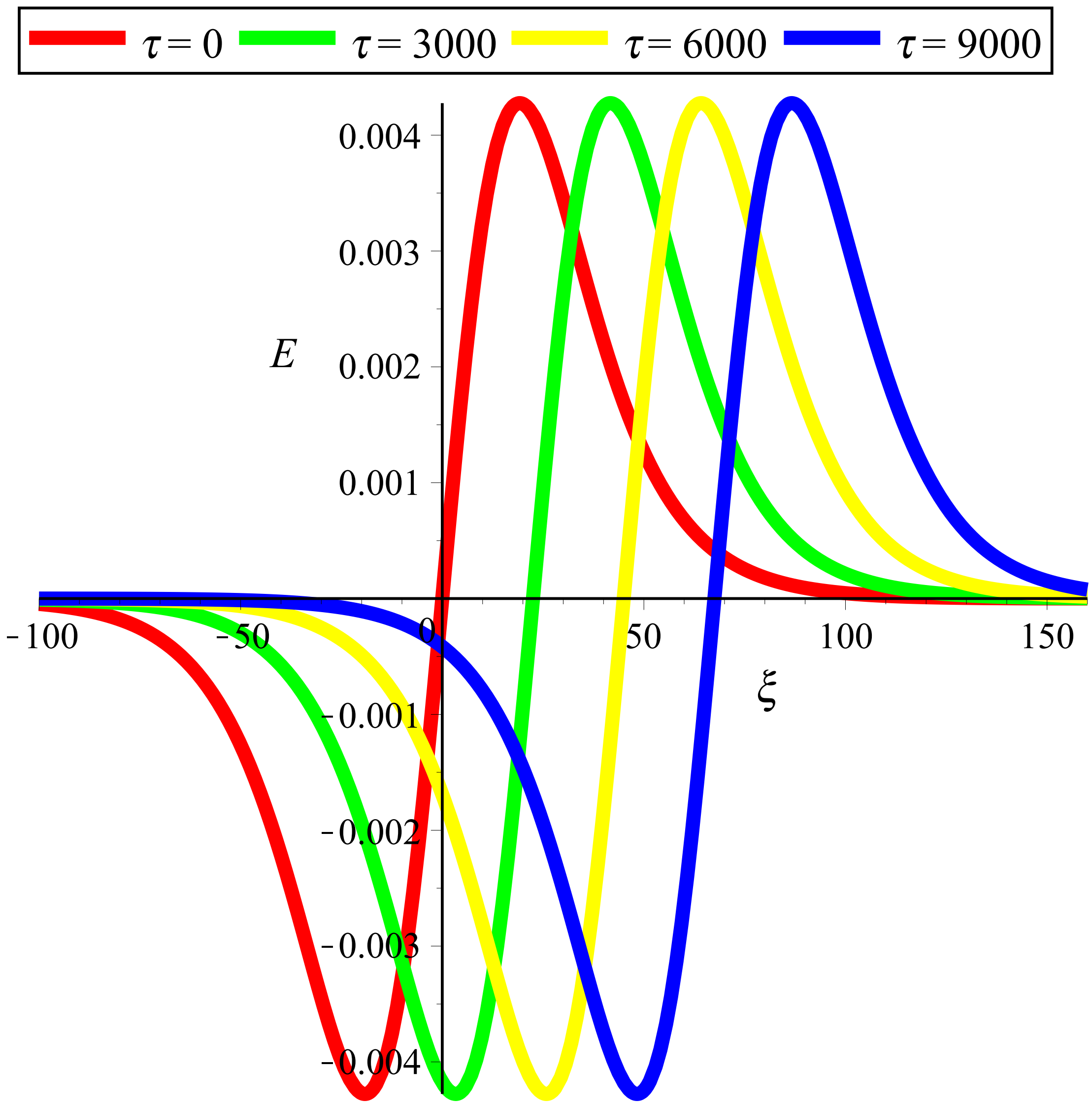

Figure 3.10: Variation of normalized electric field with regards to time. The parametric values of the parameters are considered as $\beta_{ic} = 0.1$, $\delta_{ie} = 0.1$, $\sigma = 1$, $U_0 = 0.0075$, $U_{i0} = 0.3 \times 10^8$ and $\beta_e = \beta_p = 0.7$ and $p = 0.3$.

3.5 CONCLUSIONS

A relativistic unmagnetized plasma composing of nonthermal electrons, nonthermal positrons and relativistic ion fluids has been considered to report the propagation of IASs. By implementing RPT, the KdVE has been derived and its solution has been provided. The study investigated how the plasma parameters affect the propagation characteristics of IASs, modeled by KdVE with up to 20 terms of the LRF. It was observed that in a proposed relativistic plasma environment, both compressive and rarefactive IASs are supported in the presence of nonthermality. The amplitudes and widths of IASs are significantly influenced by increasing numeric values of the plasma parameters. Moreover, the relativistic streaming factor notably alters the nonlinear propagation of IASs, with soliton energy showing a slight increase (or considerable increase) as the relativistic streaming index exceeds (or remains below) 0.1. It may be concluded that the investigations made in this article would be helpful to understand the propagation characteristics of electrostatic IASs around CVs not only in plasma sheath boundary layer of earth magnetosphere [122], Laser-plasma interaction [72,123], quark-gluon [124], interstellar medium [33,125], etc. but also in plasma laboratory [126].

Chapter 4: ION ACOUSTIC SOLITON IN AN UNMAGNETIZED RELATIVISTIC PLASMA AROUND THE CRITICAL VALUES

4.1 INTRODUCTION

Research on the electron-positron-ion (e-p-i) relativistic plasmas has become important for analyzing nonlinear wave propagation phenomena in different plasma environments, including astrophysical and laboratory applications. Since the relativistic effects considerably impact on the wave structures and dynamics of nonlinear waves as the velocity of massive charged particles approaches to the speed of light [33, 106–118]. Such types of relativistic plasmas exist in various plasma environments, e.g. space plasma phenomena [127], the plasma sheath boundary layer of Earth's magnetosphere [128], the inner region of accretion discs near black holes [27], laser-plasma interactions [72, 129], Wakefield accelerators [130], the Van Allen radiation belts [126] and so on. However, the study on the nonlinear physical phenomena in the laboratory is challenging due to the difficulty in producing such type of plasmas. On the other hand, the features of nonlinear wave propagation of such plasmas can be understood by using the mathematical tools and suitable plasma assumptions. For instance, the massive hot ions with energies from 0.1 to 100 MeV are often created in the atmosphere of the sun and interstellar medium [33, 125]. In such case, ions possess relativistic speed only when the ion energy is supported with kinetic energy. As a result, the electron-positron plasmas along with the relativistic energy of ion fluids are frequently occurred in many astrophysical and space environments, which is significant for realizing the construction and kinetics of ion acoustic waves (IAWs). The elementary features of IAWs in various e-p-i relativistic plasmas, together with the nonlinear evolution equations have already been studied in Refs. [33, 106–118]. In Ref. [114], authors have investigated the head-on collision between two IASs and propagating characteristics of IASs by deriving only the coupled Korteweg-de Vries equations (KdVEs) in the presence of nonthermal electrons and positrons. They have shown that the considered plasma is supported with both compressive and rarefactive electrostatic solitons. But, they have not provided the existence of critical values (CVs) of any specific parameter. Additionally, they have not studied what happens with the electrostatic IASs propagation around the CVs by deriving an evolution equation, that is modified KdVE (mKdVE).

To the best of our knowledge, the propagation characteristics of electrostatic solitons have not been studied by considering the weakly relativistic plasma with the existence of CVs of any specific parameter along with the remaining parameter constant. Thus, the work presented in this article will focus on the following considerations and physical issues:

(i) The unmagnetized plasma environment by the mixture of relativistic ion fluids and nonthermal distributed electrons as well as positrons.

(ii) The derivation of mKdVE with the existence of CVs of any specific parameter.

(iii) The features of nonlinear plasma wave dynamics and propagation characteristics of IASs around CVs.

4.2 FORMATION OF MKDVE

To derive the mKdVE by simplifying Eqs. (3.2) to (3.4), one can use the scaling of dependent unknown variables via the new stretched coordinates as

$$\xi = \varepsilon\left(x - V_p t\right), \ \tau = \varepsilon^3 t, 0 < \varepsilon < 1, \tag{4.1}$$

where V_p is the linear phase velocity of the perturbation mode normalized by C_s and ε measures the weakness of the dissipation. Using Eq. (4.1) into Eqs. (3.2) -(3.4), one can convert to the Eqs. (3.2) -(3.4) in the new forms involving the new stretched coordinates as

$$\varepsilon^3 \frac{\partial N_i}{\partial \tau} - \varepsilon V_p \frac{\partial N_i}{\partial \xi} + \varepsilon \frac{\partial (N_i U_i)}{\partial \xi} = 0, \tag{4.2}$$

$$\varepsilon^3 \frac{\partial \gamma U_i}{\partial \tau} - \varepsilon \frac{\partial \gamma U_i}{\partial \xi} + U_i \varepsilon \frac{\partial \gamma U_i}{\partial \xi} + \varepsilon N_i \frac{3\delta_{ie}}{1-p} \frac{\partial N_i}{\partial \xi} = -\varepsilon \frac{\partial \phi}{\partial \xi}, \tag{4.3}$$

$$\varepsilon^2 \frac{\partial^2 \phi}{\partial \xi^2} = \left(1 - \beta_e \phi + \beta_e \phi^2\right) e^{\phi} - p\left(1 + \beta_p \sigma \phi + \beta_p \sigma^2 \phi^2\right) e^{-\sigma \phi} - N_i. \tag{4.4}$$

Using the expansions of perturbed quantities N_i, U_i, and ϕ, which are involved in the power series of ε [110] from Eq. (3.9) in Eq. (4.2)-(4.4) , one can derive the different set of partial differential equations (PDEs) by taking several order of ε. The lowest order of ε PDEs obtained from the Eq. (4.2)-(4.2) are the same as in Eq. (3.10)-(3.12). Also, the first order perturbed quantities for N_i, U_i and the obtained phase velocity V_p are the same as in Eq.(3.13) and Eq.(3.14). Hence, from the next order of the ε, it gives

$$-\left(V_p - U_{i0}\right) \frac{\partial {N_i}^{(2)}}{\partial \xi} + (1-p) \frac{\partial {U_i}^{(2)}}{\partial \xi} + \frac{\partial}{\partial \xi}\left({N_i}^{(1)} {U_i}^{(1)}\right) = 0, \tag{4.5}$$

$$\begin{aligned} &-\gamma_1\left(V_p - U_{i0}\right) \frac{\partial {U_i}^{(2)}}{\partial \xi} + \frac{3\delta_{ie}}{(1-p)} \frac{\partial {N_i}^{(2)}}{\partial \xi} + \left\{\gamma_1 - \gamma_2\left(V_p - U_{i0}\right)\right\} {U_i}^{(1)} \frac{\partial {U_i}^{(1)}}{\partial \xi} \\ &+ \frac{3\delta_{ie}}{(1-p)^2} {N_i}^{(1)} \frac{\partial {N_i}^{(1)}}{\partial \xi} = -\frac{\partial \phi^{(2)}}{\partial \xi}, \end{aligned} \tag{4.6}$$

$$0 = C_1 \phi^{(2)} + C_2 \left\{\phi^{(1)}\right\}^2 - {N_i}^{(2)}, \tag{4.7}$$

where $C_2 = \left(1 - p\sigma^2\right)/2$ and $\gamma_2 = \frac{3\beta_{ic}^2}{2}$. Simplifying Eqs. (4.5) -(4.7) yield

$${N_i}^{(2)} = C_5 \left\{\phi^{(1)}\right\}^2 + \frac{(1-p)}{k} \phi^{(2)}, {U_i}^{(2)} = C_4 \left\{\phi^{(1)}\right\}^2 + \frac{\left(V_p - U_{i0}\right)}{k} \phi^{(2)}, \tag{4.8}$$

$$-\frac{\left[C_5 - C_2\right]}{2} \left\{\phi^{(1)}\right\}^2 = 0, \tag{4.9}$$

with

$$\begin{aligned} C_4 &= \tfrac{\left(V_p - U_{i0}\right)}{2k^3} \left\{\left(-\gamma_2\left(V_p - U_{i0}\right)^3 + \gamma_1\left(V_p - U_{i0}\right)^2\right) + 9\delta_{ie}\right\}, \\ C_5 &= (1-p)\left\{\tfrac{C_4}{\left(V_p - U_{i0}\right)} + \tfrac{1}{k^2}\right\}. \end{aligned} \tag{4.10}$$

Eq. (13) yields $A_c = \left[C_5 - C_2\right]$ must be zero because $\phi^{(1)} \neq 0$, which is equal to the non-linear coefficient of KdVE. It is clearly provided that one can determine the CVs of any specific parameter (say, temperature ratio's, density ratio's, etc) to study the soliton propa-

Table 4.1: List of some CVs for density ratio's

Parametric values	CVs (say p_c) for density ratio's ($p = N_{p0}/N_{e0}$)
$\beta_e = \beta_p = 0.7$, $\beta = 0.05$, $\sigma = 1$, $\delta = 0.1$	$p_c = 0.2741291620$
$\beta_e = \beta_p = 0.7$, $\beta = 0.05$, $\sigma = 2.5$, $\delta = 0.1$	$p_c = 0.08255322433$
$\beta_e = \beta_p = 0.5$, $\beta = 0.1$, $\sigma = 1$, $\delta = 0.1$	$p_c = 0.03467135200$
$\beta_e = \beta_p = 0.5$, $\beta = 0.1$, $\sigma = 1$, $\delta = 0.01$	$p_c = 0.07681842699$

gation around CVs by setting $A_c = 0$. The existence of some CVs is shown in Table 4.1.

Finally, the next order of ε PDEs are derived as follows:

$$\frac{\partial {N_i}^{(1)}}{\partial \tau} - (V_p - U_{i0})\frac{\partial {N_i}^{(3)}}{\partial \xi} + (1-p)\frac{\partial {U_i}^{(3)}}{\partial \xi} + \frac{\partial}{\partial \xi}\left({N_i}^{(1)}{U_i}^{(2)} + {N_i}^{(2)}{U_i}^{(1)}\right) = 0, \quad (4.11)$$

$$\begin{aligned}\gamma_1\frac{\partial U_i^{(1)}}{\partial \tau} - (V_p - U_{i0})\,\gamma_1\frac{\partial U_i^{(3)}}{\partial \xi} + \frac{3\delta_{ie}}{1-p}\frac{\partial N_i^{(3)}}{\partial \xi} + \left\{\gamma_1 - \gamma_2\,(V_p - U_{i0})\right\}\frac{\partial\left(U_i^{(2)}U_i^{(1)}\right)}{\partial \xi} + \\ \gamma_2\left\{U_i^{(1)}\right\}^2\frac{\partial U_i^{(1)}}{\partial \xi} + \frac{3\delta_{ie}}{(1-p)^2}\frac{\partial (N_i^{(1)}N_i^{(2)})}{\partial \xi} = -\frac{\partial \phi^{(3)}}{\partial \xi}\end{aligned} \quad (4.12)$$

$$\frac{\partial^2 \phi^{(1)}}{\partial \xi^2} = C_1\phi^{(3)} + 2C_2\phi^{(1)}\phi^{(2)} + C_3\left\{\phi^{(1)}\right\}^3 - {N_i}^{(3)}\ , \quad (4.13)$$

where $C_3 = \left[(1+3\beta_e) + p\sigma^3\,(1+3\beta_p)\right]/6$. Combining Eqs. (4.10) -(4.12) with the help of Eqs. (3.13), (4.8) and (4.9), the following mKdVE is obtained:

$$\frac{\partial \phi^{(1)}}{\partial \tau} + A\left\{\phi^{(1)}\right\}^2\frac{\partial \phi^{(1)}}{\partial \xi} + B\frac{\partial^3 \phi^{(1)}}{\partial \xi^3} = 0, \quad (4.14)$$

where

$$\begin{aligned}A = \frac{k}{2\gamma_1(1-p)}\Bigg[-\frac{3C_3k}{(V_p-U_{i0})} + \frac{1}{k}\left\{3C_4(1-p)(2\gamma_1 - \gamma_2\,(V_p - U_{i0})) + \frac{3C_5}{(V_p-U_{i0})}(k+6\delta_{ie})\right\} \\ + \left(\frac{\gamma_2(1-p)}{k^3}\right)(V_p - U_{i0})^2\Bigg]\end{aligned}$$

$$B = \frac{k^2}{2\gamma_1\,(V_p - U_{i0})\,(1-p)}.$$

The well-established soliton solution of Eq. (17) is obtained by considering a reference frame $\zeta = \xi - U_0\tau$ (U_0 stands for the constant reference speed) and $\phi^{(1)} \to 0$, $\frac{d\phi^{(1)}}{d\zeta} \to 0$, ...as $\zeta \to \pm\infty$ in the following form:

$$\phi^{(1)} = \phi_a sech\left\{\frac{\zeta}{\phi_w}\right\}, \tag{4.15}$$

where $\phi_a = \left(\sqrt{6U_0/A}\right)$ and $\phi_w = \sqrt{(B/U_0)}$ are the amplitudes and widths of the soliton, respectively.

4.3 RESULTS AND DISCUSSIONS

In this section, the effect of $p(= N_{p0}/N_{e0})$, $\delta_{ie}(= T_i/T_e)$, $\sigma(= T_e/T_p)$, $\beta = \beta_e = \beta_p$ (population of nonthernal electrons and positrons) and $\beta_{ic}(= U_{i0}/c)$ on the propagation features of electrostatic IASs has been analyzed by using the solution of mKdVE. In this analysis, the parametric values of the parameters are considered based on the Refs. [33, 107–114], which are relevant to some astrophysical and space environment [33, 122–125].

Figures 1(a), 1(b), 1(c) and 1(d) display the impact of p, δ_{ie}, σ and β on the electrostatic soliton propagation around their CVs in the considered weakly relativistic plasma with the choice of the remaining parameter constant. Whereas, Figures 2(a), 2(b), 2(c) and 2(d) display the impact of p, δ_{ie}, σ and β on the normalized electric field $\mathbf{E} = -\nabla\phi^{(1)}$ around their CVs with the choice of the reaming parameter constant. It is observed from Figure 1 and 2 that the electrostatic IASs propagation and their corresponding normalized electric fields are supported around CVs if p, δ_{ie} and σ are greater than their CVs p_c, δ_c and σ_c and $\beta = \beta_e = \beta_p$ is less than its CV β_c. Otherwise, the electrostatic IASs propagation and their corresponding normalized electric fields cannot be supported due to the complex parametric values of the nonlinear coefficient (A) of mKDVE. As a result, mKDVE is applicable to analyze the electrostatic wave propagation in unmodulated region of space. Figures 1 and 2 also depicted that the amplitude and width of electrostatic IASs and their corresponding normalized electric fields are increasing (decreasing) with the increase of β (p,δ_{ie} and σ) around the CVs. Additionally, the considered plasma environment is only supported with the compressive electrostatic potential around CVs. Such electrostatic potential produces the bell-shaped (soliton) type structures, whereas the normalized electric field produces the semi-kink shaped type structures in the plasma environment. Finally, Figures 3(a) and 3(b) display the electrostatic potential and the normalized electric field with the variation of time. It is clearly found that the electrostatic potential and the normalized electric field are possessed the pulse like structures, as it is expected. Thus, the work has been made in this articles would be helpful to understand the nonlinear propagation features of IASs in the presence of relativistic ion fluids and nonthermal electrons as well as positrons in many astrophysical and space environments.

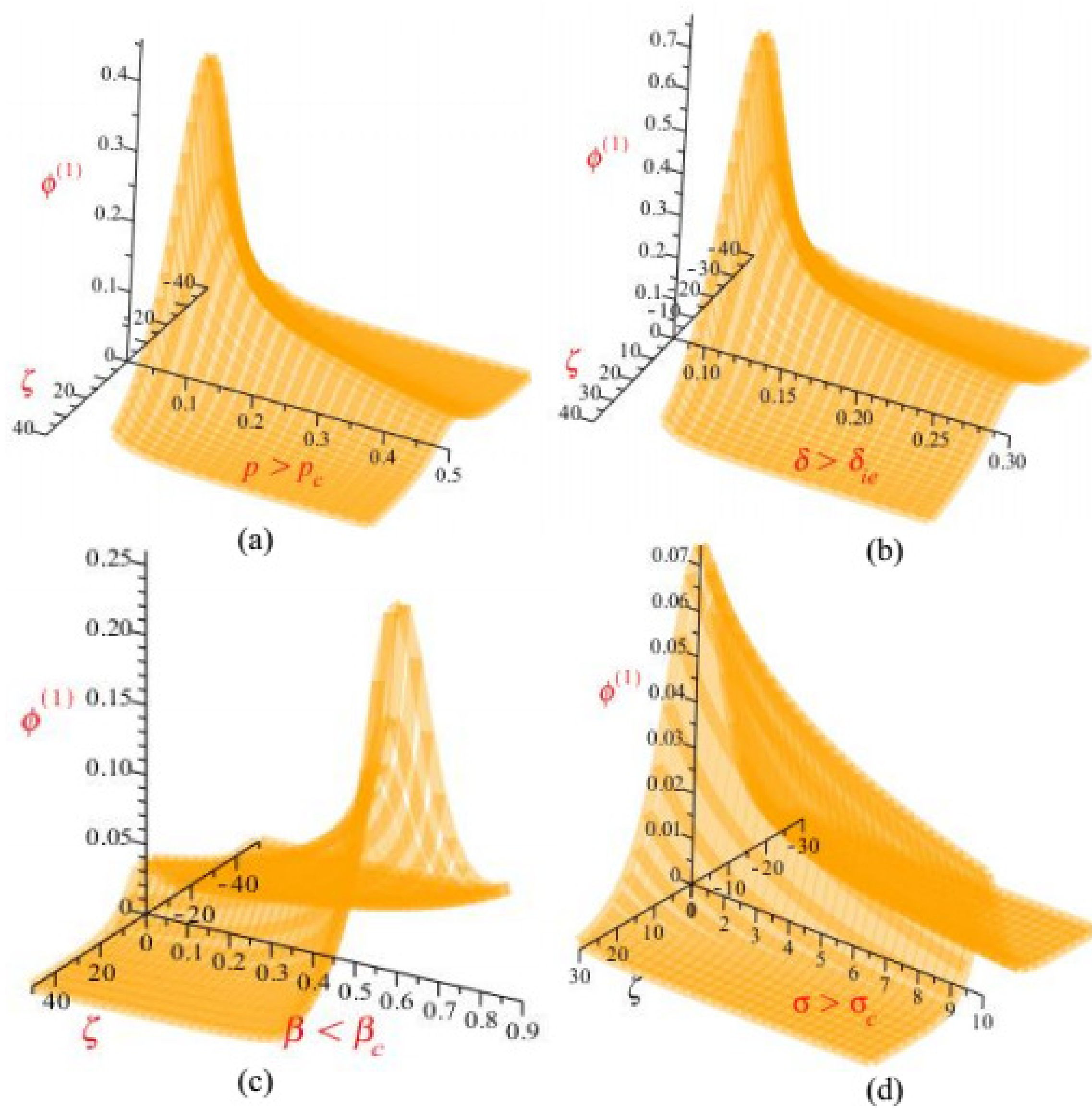

Figure 4.1: Electrostatic potential with regards to (a) p and ζ with $\beta = 0.5$, $\delta_{ie} = 0.1$, $\sigma = 1$, $U_0 = 0.0075$, $U_{i0} = 0.3 \times 10^8$ and $\beta_{ic} = 0.1$; (b) δ_{ie} and ζ with $\beta = 0.5$, $p = 0.07681842699$, $\sigma = 1$, $U_0 = 0.0075$, $U_{i0} = 0.3 \times 10^8$ and $\beta_{ic} = 0.1$; (c) $\beta_e = \beta_p = \beta$ and ζ with $p = 0.2741291620$, $\delta_{ie} = 0.1$, $\sigma = 1$, $U_0 = 0.0075$, $U_{i0} = 0.15 \times 10^8$ and $\beta_{ic} = 0.05$; and (d) σ and ζ with $p = 0.3$, $\delta_{ie} = 0.1$, $\beta = 0.1$, $U_0 = 0.0075$, $U_{i0} = 0.03 \times 10^8$ and $\beta_{ic} = 0.01$.

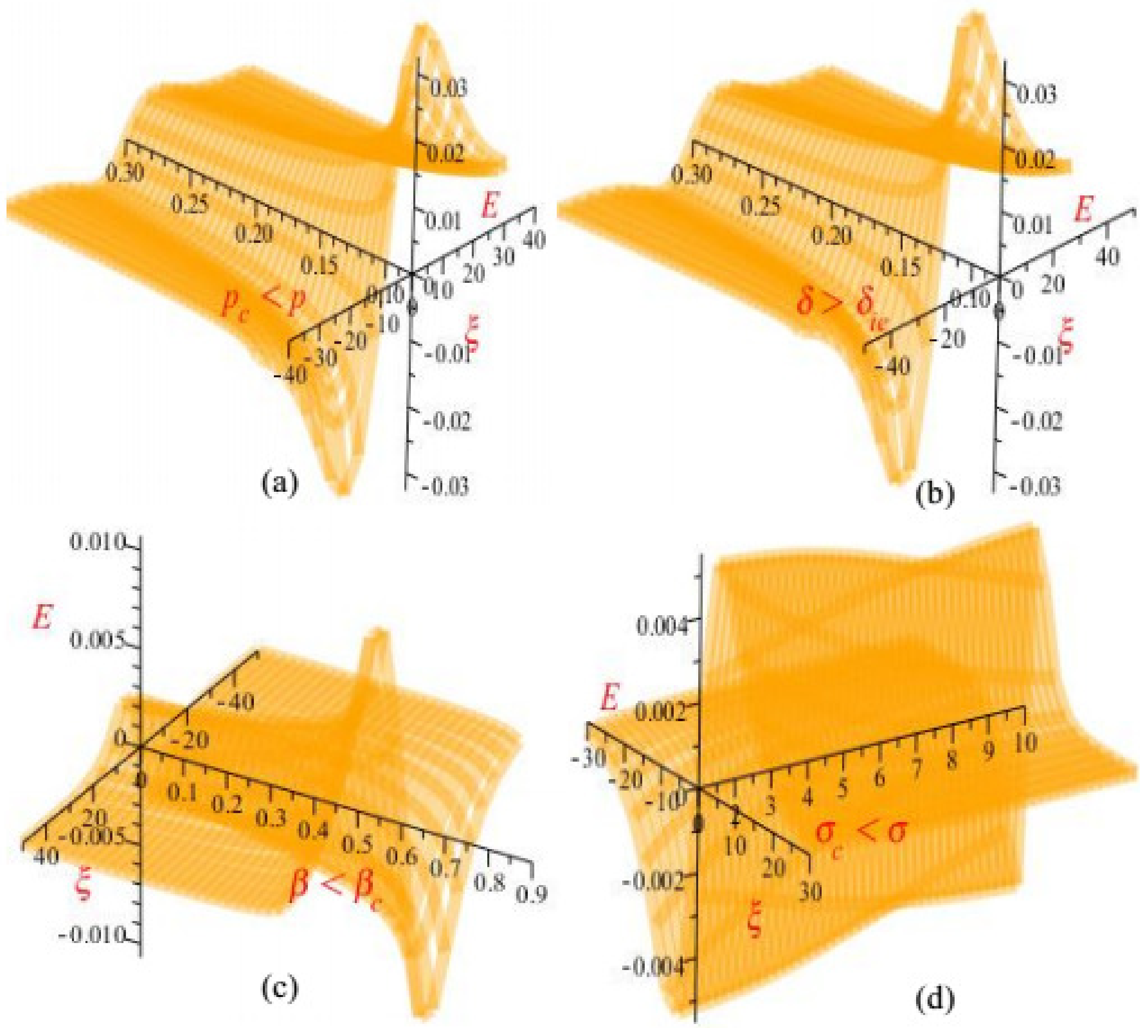

Figure 4.2: The normalized electric field with regards to (a) p and ζ with $\beta = 0.5$, $\delta_{ie} = 0.1$, $\sigma = 1$, $U_0 = 0.0075$, $U_{i0} = 0.3 \times 10^8$ and $\beta_{ic} = 0.1$; (b) δ_{ie} and ζ with $\beta = 0.5$, $p = 0.07681842699$, $\sigma = 1$, $U_0 = 0.0075$, $U_{i0} = 0.3 \times 10^8$ and $\beta_{ic} = 0.1$; (c) $\beta_e = \beta_p = \beta$ and ζ with $p = 0.2741291620$, $\delta_{ie} = 0.1$, $\sigma = 1$, $U_0 = 0.0075$, $U_{i0} = 0.15 \times 10^8$ and $\beta_{ic} = 0.05$; and (d) σ and ζ with $p = 0.3$, $\delta_{ie} = 0.1$, $\beta = 0.1$, $U_0 = 0.0075$, $U_{i0} = 0.03 \times 10^8$ and $\beta_{ic} = 0.01$.

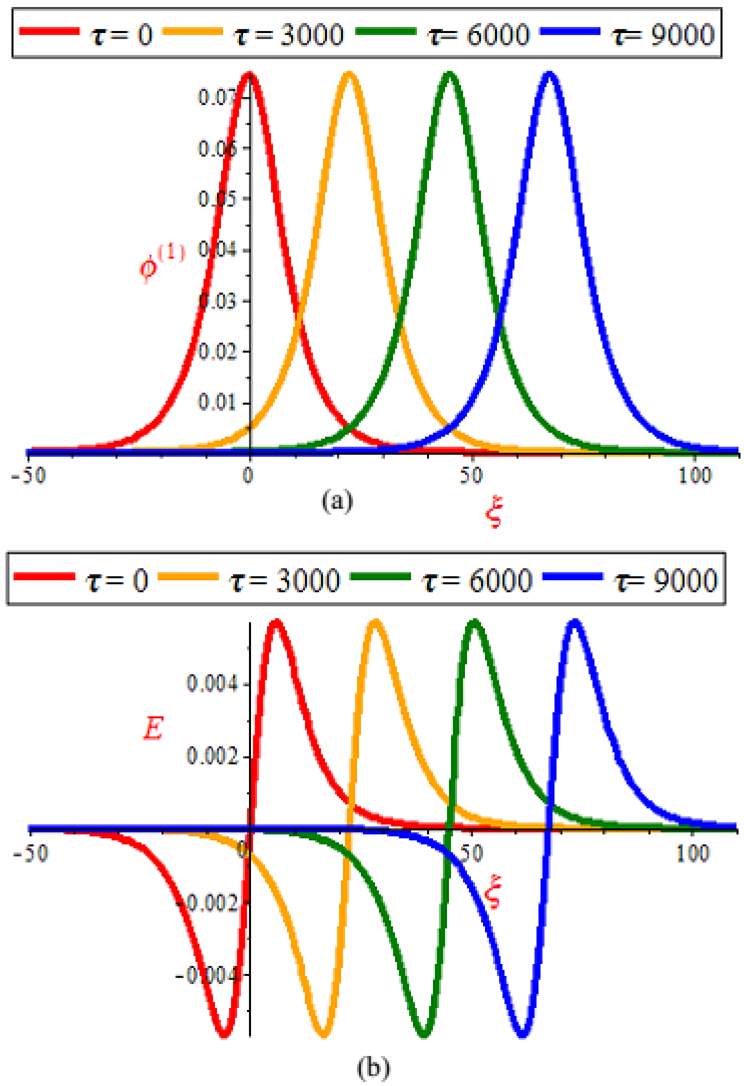

Figure 4.3: Variation of (a) electrostatic potential and (b) normalized electric field with regards to time. The parametric values of the parameters are considered as $\beta = 0.1$, $\delta_{ie} = 0.1$, $\sigma = 1$, $U_0 = 0.0075$, $U_{i0} = 0.03 \times 10^8$ and $\beta_{ic} = 0.01$ and $p = 0.3$.

4.4 CONCLUSIONS

An unmagnetized plasma composing of nonthermal electrons, nonthermal positrons and relativistic ion fluids has been considered to report the propagation of IASs around the CVs for weakly relativistic regime. By implementing the reductive perturbation method, the mKdVE has been derived with the correction of the stretching coordinates. The impacts of plasma parameters on the properties of electrostatic IASs and their corresponding electric fields have been investigated by the soliton solutions of mKdVE. It has been found that (i) the considered plasma environment supports the CVs and (ii) the compressive electrostatic soliton propagation exists around the CVs. The electrostatic IASs are formed bell-shaped type structures, whereas the corresponding normalized electric field is formed semi-kink shaped type structures. It is also found that the amplitudes and widths of IASs are increased (decreased) with the increase of relativistic streaming factor (density and temperature ratio,s). It may be concluded that the investigations made in this article would be helpful to understand the propagation characteristics of electrostatic IASs around CVs not only in plasma sheath boundary layer of earth magnetosphere [122], Laser-plasma interaction [72, 123], quark-gluon [124], interstellar medium [33, 125], etc. but also in plasma laboratory [126].

Chapter 5: ION ACOUSTIC SOLITON IN AN UNMAGNETIZED RELATIVISTIC PLASMA AROUND THE SUPER-CRITICAL VALUES

5.1 INTRODUCTION

The aim of this chapter is to expand upon the IASs propagation for super-critical values (SCVs) introduced in the chapters 3 and 4 by deriving a new evolution equation with quartic nonlinearity. Key questions addressed include: (i) determining the existence of SCVs with specific parameters in the considered model equations for UMRPs , (ii) identifying the appropriate evolution equation for studying IASs propagation in this context, (iii) examining the nature of electrostatic IASs propagation not only around SCVs but also at the SCVs themselves and (iv) analyzing how plasma parameters influence the existence regions. These findings are anticipated to enhance our understanding of nonlinear electrostatic IASs propagation not only around SCVs but also at the SCVs in interplanetary and astrophysical plasmas, as well as in laboratory settings.

5.2 FORMATION OF KDVE WITH QUARTIC NONLINEARITY AND STATIONARY SOLITON SOLUTION

To derive the KdVE involving more higher order nonlinearity, by simplifying Eqs. (3.2) to (3.4), one can use the scaling of dependent unknown variables via the new stretched coordinates as

$$\xi = \varepsilon^{\frac{3}{2}}(x - V_p t), \ \tau = \varepsilon^{\frac{9}{2}} t, 0 < \varepsilon < 1, \tag{5.1}$$

where V_p is the linear phase velocity of the perturbation mode normalized by C_s and ε measures the weakness of the dissipation. Using Eq. (5.1) into Eqs. (3.2) -(3.4), one can convert to the Eqs. (3.2) -(3.4) in the new forms involving the new stretched coordinates as

$$\varepsilon^{9/2}\frac{\partial N_i}{\partial \tau} - \varepsilon^{3/2} V_p \frac{\partial N_i}{\partial \xi} + \varepsilon^{3/2}\frac{\partial (N_i U_i)}{\partial \xi} = 0, \tag{5.2}$$

$$\varepsilon^{9/2}\frac{\partial\gamma U_i}{\partial\tau}-\varepsilon^{3/2}\frac{\partial\gamma U_i}{\partial\xi}+U_i\varepsilon^{3/2}\frac{\partial\gamma U_i}{\partial\xi}+\varepsilon^{3/2}N_i\frac{3\delta_{ie}}{1-p}\frac{\partial N_i}{\partial\xi}=-\varepsilon^{3/2}\frac{\partial\phi}{\partial\xi}, \tag{5.3}$$

$$\varepsilon^3\frac{\partial^2\phi}{\partial\xi^2}=\left(1-\beta_e\phi+\beta_e\phi^2\right)e^{\phi}-p\left(1+\beta_p\sigma\phi+\beta_p\sigma^2\phi^2\right)e^{-\sigma\phi}-N_i. \tag{5.4}$$

Using the expansions of perturbed quantities N_i, U_i, and ϕ as in Eq. (3.9) in to Eq. (5.2)-(5.4), one can derive the different set of PDEs by taking several order of ε. The lowest order of ε PDEs obtained from the Eq. (5.2)-(5.4) are the same as in Eq. (3.10)-(3.12). Also, the first order perturbed quantities for N_i, U_i and the obtained phase velocity V_p are the same as in Eq.(3.13) and Eq.(3.14). Again, from the next order of ε the obtained PDEs yield

$$N_i^{(2)}=C_5\left\{\phi^{(1)}\right\}^2+\frac{(1-p)}{k}\phi^{(2)},U_i^{(2)}=C_4\left\{\phi^{(1)}\right\}^2+\frac{(V_p-U_{i0})}{k}\phi^{(2)}, \tag{5.5}$$

$$-\frac{[C_5-C_2]}{2}\left\{\phi^{(1)}\right\}^2=0, \tag{5.6}$$

with

$$\begin{gathered}C_4=\frac{(V_p-U_{i0})}{2k^3}\left\{\left(-\gamma_2\left(V_p-U_{i0}\right)^3+\gamma_1\left(V_p-U_{i0}\right)^2\right)+9\delta_{ie}\right\},\\ C_5=(1-p)\left\{\frac{C_4}{\left(V_p-U_{i0}\right)}+\frac{1}{k^2}\right\}.\end{gathered} \tag{5.7}$$

Eq. (5.6) yields $A_c=[C_5-C_2]$ must be zero because $\phi^{(1)}\neq 0$, which is similar to the nonlinear coefficient (P) of KdVE. It is provided that one can easily determine the CVs of any one parameter with the reaming parameter constant by setting $P=0$.

Now, the next order of the ε yields,

$$-\left(V_p-U_{i0}\right)\frac{\partial N_i^{(3)}}{\partial\xi}+\frac{\partial N_i^{(2)}U_i^{(1)}}{\partial\xi}+\frac{\partial N_i^{(1)}U_i^{(2)}}{\partial\xi}=0, \tag{5.8}$$

$$\begin{gathered}-\left(V_p-U_{i0}\right)\gamma_1\frac{\partial U_i^{(3)}}{\partial\xi}+\frac{3\delta_{ie}}{1-p}\frac{\partial N_i^{(3)}}{\partial\xi}+\left\{\gamma_1-\gamma_2\left(V_p-U_{i0}\right)\right\}\frac{\partial\left(U_i^{(2)}U_i^{(1)}\right)}{\partial\xi}+\\ \gamma_2\left\{U_i^{(1)}\right\}^2\frac{\partial U_i^{(1)}}{\partial\xi}+\frac{3\delta_{ie}}{(1-p)^2}\frac{\partial(N_i^{(1)}N_i^{(2)})}{\partial\xi}=-\frac{\partial\phi^{(3)}}{\partial\xi}\end{gathered} \tag{5.9}$$

$$0=C_3{\phi^{(1)}}^3+2C_2\phi^{(1)}\phi^{(2)}+C_1\phi^{(3)}-N_i^{(3)}. \tag{5.10}$$

Simplifying Eqs. (5.7) -(5.9) yield

$$\begin{aligned} N_i^{(3)} &= C_7 \left\{\phi^{(1)}\right\}^3 + 2C_5 \phi^{(1)}\phi^{(2)} + \tfrac{(1-p)}{k}\phi^{(3)}, \\ U_i^{(3)} &= C_6 \left\{\phi^{(1)}\right\}^3 + 2C_4 \phi^{(1)}\phi^{(2)} + \tfrac{\left(V_p-U_{i0}\right)}{k}\phi^{(3)}, \end{aligned} \tag{5.11}$$

$$-\frac{[C_7 - C_3]}{2}\left\{\phi^{(1)}\right\}^3 = 0, \tag{5.12}$$

with

$$C_6 = \tfrac{C_4}{k^2}\left\{(\gamma_1 - (V_p - U_{i0})\gamma_2)(V_p - U_{i0})^2 + 3\delta_{ie}\right\} + \tfrac{6\delta_{ie}C_5\left(V_p-U_{i0}\right)}{k^2(1-p)} + \tfrac{\gamma_2}{3k^4}(V_p - U_{i0})^4,$$
$$C_7 = \tfrac{(1-p)(C_4+kC_6)+C_5\left(V_p-U_{i0}\right)}{k\left(V_p-U_{i0}\right)}.$$

Eq. (5.11) yields $A_{sc} = [C_7 - C_3]$ must be zero because $\phi^{(1)} \neq 0$, which is similar to the nonlinear coefficient (A) of mKdVE. It is provided that one can easily determine the SCVs of any one parameter with the reaming parameter constant by setting together with $P = 0$ and $A = 0$. Finally, the next order equations are determined in the following forms:

$$\begin{aligned} -(V_p - U_{i0})\frac{\partial N_i^{(4)}}{\partial \xi} + (1-p)\frac{\partial U_i^{(4)}}{\partial \xi} + \frac{\partial N_i^{(1)}}{\partial \tau} + \frac{\partial (U_i^{(3)}N_i^{(1)})}{\partial \xi} + \\ \frac{\partial (N_i^{(3)}U_i^{(1)})}{\partial \xi} + \frac{\partial (N_i^{(2)}U_i^{(2)})}{\partial \xi} = 0, \end{aligned} \tag{5.13}$$

$$\begin{aligned} -(V_p - U_{i0})\frac{\partial U_i^{(4)}}{\partial \xi} + \tfrac{3\delta_{ie}}{(1-p)}\frac{\partial N_i^{(4)}}{\partial \xi} + \frac{\partial \phi^{(4)}}{\partial \xi} + (\gamma_1 - \gamma_2(V_p - U_{i0}))\frac{\partial (U_i^{(1)}U_i^{(3)})}{\partial \xi} + \\ (\gamma_1 - \gamma_2(V_p - U_{i0}))U_i^{(2)}\frac{\partial U_i^{(2)}}{\partial \xi} + \gamma_2\left\{U_i^{(1)}\right\}^2\frac{\partial U_i(2)}{\partial \xi} + \\ \tfrac{3\delta_{ie}}{(1-p)^2}\frac{\partial (N_i^{(1)}N_i^{(3)})}{\partial \xi} + \tfrac{3\delta_{ie}}{(1-p)^2}N_i^{(2)}\frac{\partial N_i^{(2)}}{\partial \xi} + \gamma_1\frac{\partial U_i^{(1)}}{\partial \tau} = 0, \end{aligned} \tag{5.14}$$

$$0 = Z_4\left\{\phi^{(1)}\right\}^4 + 3C_3 + 2C_2\phi^{(1)}\phi^{(2)} + C_1\phi^{(3)} - N_i^{(3)}. \tag{5.15}$$

where $Z_4 = \left[(1+8\beta_e) - p\sigma^4(1+8\beta_p)\right]/24$. Combining Eqs. (5.13) -(5.15) with the help of Eqs. (3.13), (5.5) and (5.11), the following quartic KdVE is obtained:

$$\frac{\partial \phi^{(1)}}{\partial \tau} + R\left\{\phi^{(1)}\right\}^3\frac{\partial \phi^{(1)}}{\partial \xi} + S\frac{\partial^3 \phi^{(1)}}{\partial \xi^3} = 0, \tag{5.16}$$

where

$$R=\frac{1}{\gamma_1}\left[\frac{k}{(1-p)^2\left(V_p-U_{i0}\right)}\left\{(1-p)^2C_4^2\left(\gamma_1-\gamma_2\left(V_p-U_{i0}\right)\right)+2C_4C_5\gamma_1(1-p)\left(V_p-U_{i0}\right)\right.\right.$$
$$\left.-2k(1-p)Z_4+3\delta_{ie}C_5^2\right\}+\frac{2}{(1-p)\left(V_p-U_{i0}\right)}\left\{(1-p)C_6\left(-\gamma_2\left(V_p-U_{i0}\right)^2+2\gamma_1\left(V_p-U_{i0}\right)\right)+\right.$$
$$\left.\left.C_7\left(k+6\delta_{ie}\right)\right\}+\frac{2C_4\gamma_2}{k}\left(V_p-U_{i0}\right)\right]$$

$$S=\frac{k^2}{2\gamma_1\left(V_p-U_{i0}\right)(1-p)}.$$

The well-established soliton solution of Eq. (5.16) is obtained by considering a reference frame $\zeta=\xi-U_0\tau$ (U_0 stands for the constant reference speed) and $\phi^{(1)}\rightarrow 0$, $\frac{d\phi^{(1)}}{d\zeta}\rightarrow 0$, ...as $\zeta\rightarrow\pm\infty$ in the following form:

$$-U_0\phi^{(1)}+\frac{R}{4}\left\{\phi^{(1)}\right\}^4+S\frac{d^2\phi_1}{d\chi^2}=0 \tag{5.17}$$

Eq. (5.17) can be represent in planar dynamical system and the dynamical system (5.17) is a Hamiltonian system with Hamiltonian function

$$H(\phi^{(1)},z)=\frac{z^2}{2}-\frac{U_0}{2S}\left\{\phi^{(1)}\right\}^2+\frac{R}{10S}\left\{\phi^{(1)}\right\}^5. \tag{5.18}$$

For any homoclinic orbit of the dynamical system (5.17) at $(0,0)$, one can have $H(\phi^{(1)},z)=0$, which gives

$$\frac{d\phi^{(1)}}{\phi^{(1)}\sqrt{1-\frac{R}{10U_0}\left\{\phi^{(1)}\right\}^3}}=\pm\sqrt{\frac{U_0}{S}}d\chi. \tag{5.19}$$

Let $\sqrt{1-\frac{R}{10U_0}\left\{\phi^{(1)}\right\}^3}=f^2$, applying this in equation (5.19) and by integrating we obtain

$$f=sech\left(\pm\frac{3}{2}\sqrt{\frac{U_0}{S}}\chi\right) \tag{5.20}$$

Using f, one can obtain

$$\phi^{(1)} = \phi_m sech^{\frac{2}{3}} \left\{ \frac{\chi}{W} \right\}. \tag{5.21}$$

Equation (5.21) represents the solitary wave solution of the quartic KdVE where $\phi_m = \left(\frac{10U_0}{R}\right)^{\frac{1}{3}}$ and $W = \sqrt{\frac{4S}{9U_0}}$ are the amplitude and width of the soliton, respectively.

5.3 RESULTS AND DISCUSSIONS

In this section, we discuss the propagation characteristics of small-amplitude nonlinear IASs, considering the influence of plasma parameters through an analysis of the soliton solution of the KdVE with quartic nonlinearity. The parameter values are assumed based on reference [40], relevant to astrophysical and space environments. It is noted that the quartic nonlinearity in the KdVE emerges when the nonlinear coefficients of both KdVE and mKdVE are set to zero, defining SCVs. Our study confirms the existence of these SCVs in the plasma environment under consideration. The key findings based on these assumptions are detailed below.

Figure 5.1 depict the appearance of SCV p_{SC} for the density ratio p, respectively with presence of nonthermality, alongside constant values of other parameters. This figure illustrates how the nonlinear coefficients P and A of the KdVE and mKdVE become zero at p_{SC}, allowing us to study the propagation of IASs around this point. Subsequently, Figures 5.2 to 5.5 explore the effects of p, δ_{ie}, σ, and p on the nonlinear propagation of IASs in relativistic plasma, considering the LRF up to 20 terms and holding other parameters constant. The analysis reveals that the plasma supports finite-amplitude soliton structures whose amplitude, and width are strongly influenced by these parameters. Figures 5.2, 5.3 and 5.5 reveal that the amplitude and width of IASs decrease with the increasing electron-to-positron temperature ratio σ, ion-to-electron temperature ratio δ_{ie}, and positron-to-electron density ratio p. However, Figure 5.4 illustrates the increase of the width and amplitude of the IASs with increasing value of β.

Figure 5.7 shows the effect of the electric field on the propagation characteristics of IASs decreases by the increase of ion-to-electron temperature ratio δ_{ie}. Whereas, from Figure 5.6, 5.8 and 5.9 it is found that the effect of the electric field decreases (increases) with the increase of electron-to-positron temperature ratio σ and positron-to-electron density ratio p (distributive parameter β) up to numeric value 0.6 and then gains a slight increase.

Finally, Figures 5.10 and 5.11 that both electrostatic potential and normalized electric field are behaves pulse like shaped with the increase of time. It is also clearly found that the electrostatic potential and the normalized electric field are possessed the pulse like structures, as it is expected. It is noted that the expansion of LRF are considered up to 20 terms in this presented studies.

In conclusion, this study suggests that to further understand the nonlinear propagation of IASs in relativistic plasmas, researchers should consider extending the LRF to higher-order terms and exploring higher values of the relativistic streaming index β_{ic}. These insights are crucial not only for theoretical advancements but also for practical applications in various plasma physics contexts, including laser-plasma interactions, quark-gluon environments, dark matter studies, solar atmospheres, and laboratory experiments.

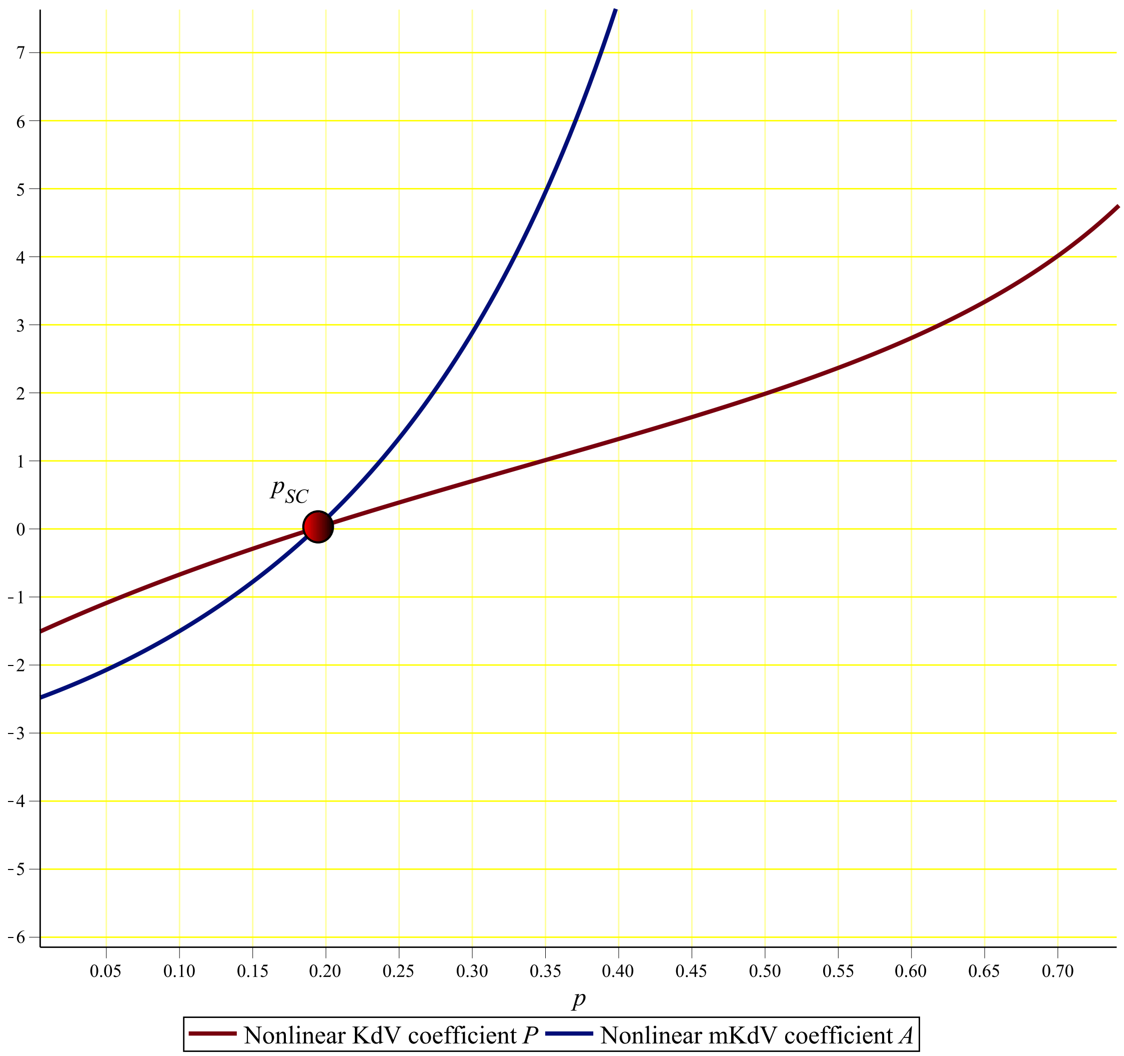

Figure 5.1: Super-critical Value (SCV) regards to p with $\beta = 0.7$, $\delta_{ie} = 0.4$, $\sigma = 1$, $U_0 = 0.0075$, $U_{i0} = 0.3 \times 10^8$ and $\beta_{ic} = 0.1$.

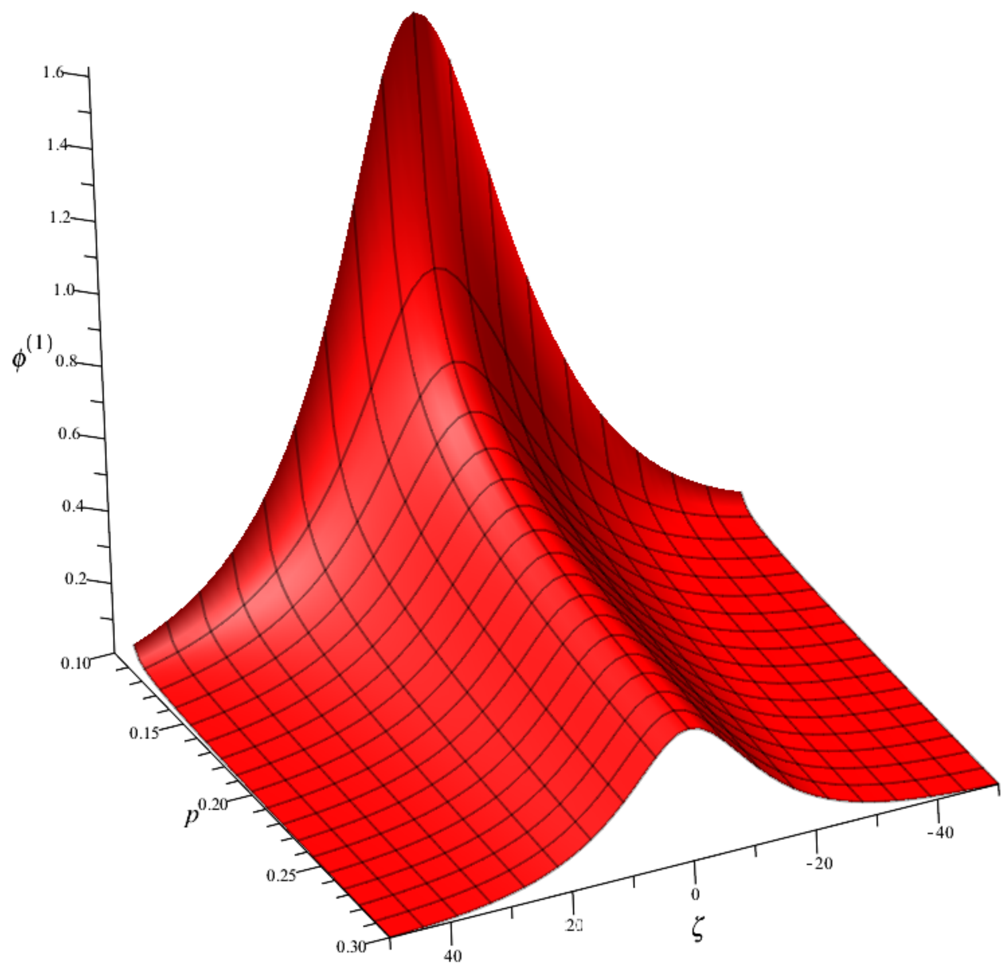

Figure 5.2: Electrostatic potential with regards to p and ζ with $\beta = 0.7$, $\delta_{ie} > \delta_{SC} = 0.5$, $\sigma = 1.5$, $U_0 = 0.0075$, $U_{i0} = 0.9 \times 10^8$ and $\beta_{ic} = 0.3$.

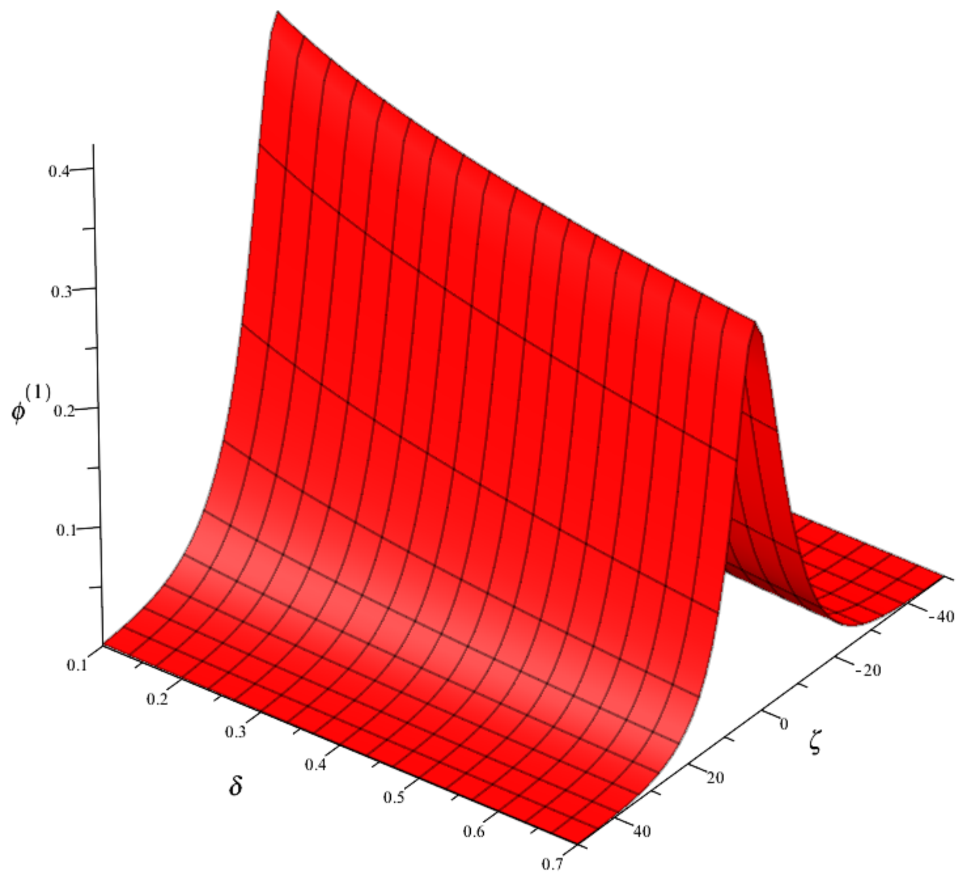

Figure 5.3: The Electrostatic potential with regards to δ_{ie} and ζ with $\beta = 0.5$, $p > p_{SC} = 0.3$, $\sigma = 1$, $U_0 = 0.0075$, $U_{i0} = 1.5 \times 10^8$ and $\beta_{ic} = 0.5$.

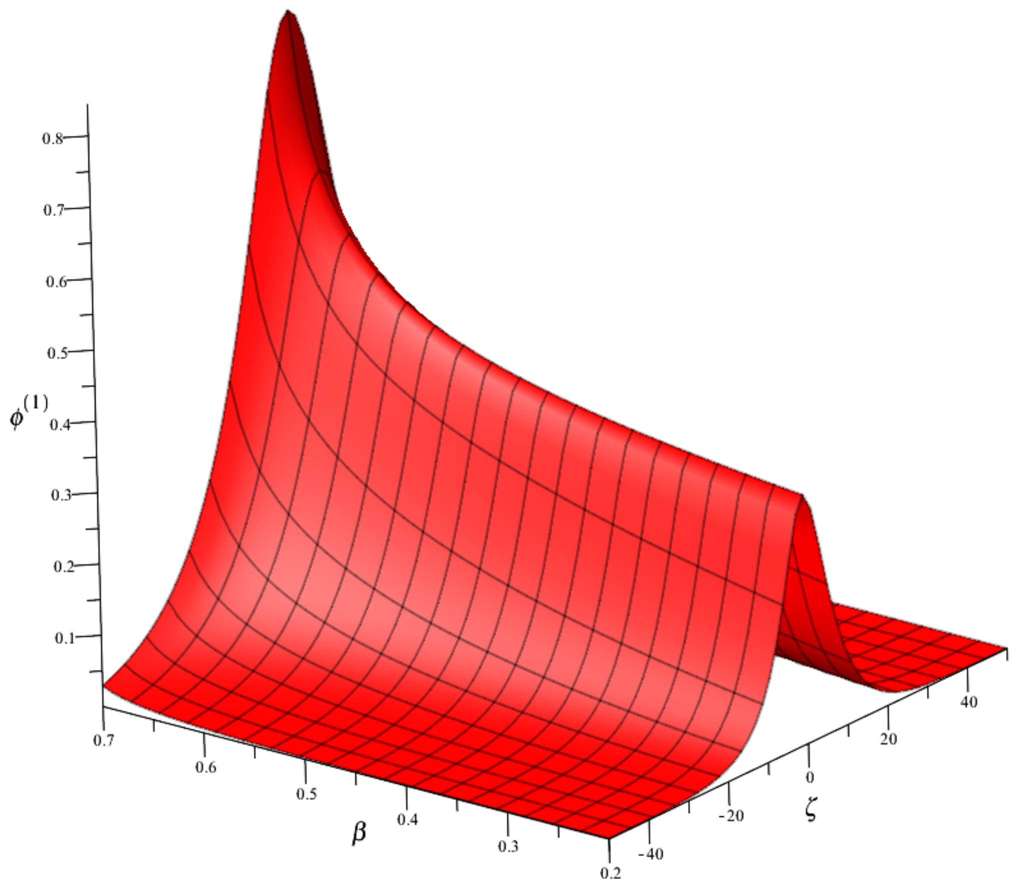

Figure 5.4: The Electrostatic potential with regards to $\beta_e = \beta_p = \beta$ and ζ with $p > p_{SC} = 0.3$, $\delta_{ie} = 0.1$, $\sigma = 1$, $U_0 = 0.0075$, $U_{i0} = 1.5 \times 10^8$ and $\beta_{ic} = 0.5$.

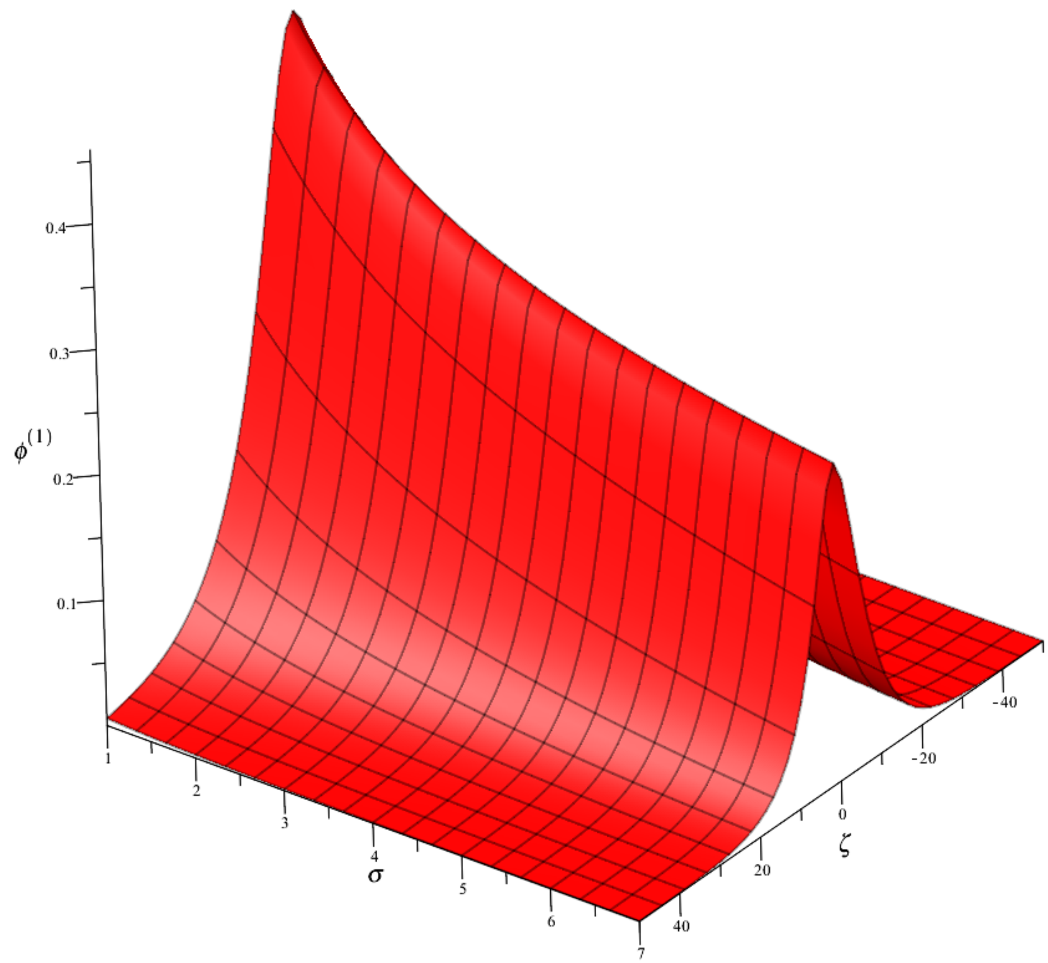

Figure 5.5: The Electrostatic potential with regards to σ and ζ with $p > p_{SC} = 0.2$, $\delta_{ie} = 0.1$, $\beta = 0.5$, $U_0 = 0.0075$, $U_{i0} = 0.9 \times 10^8$ and $\beta_{ic} = 0.3$.

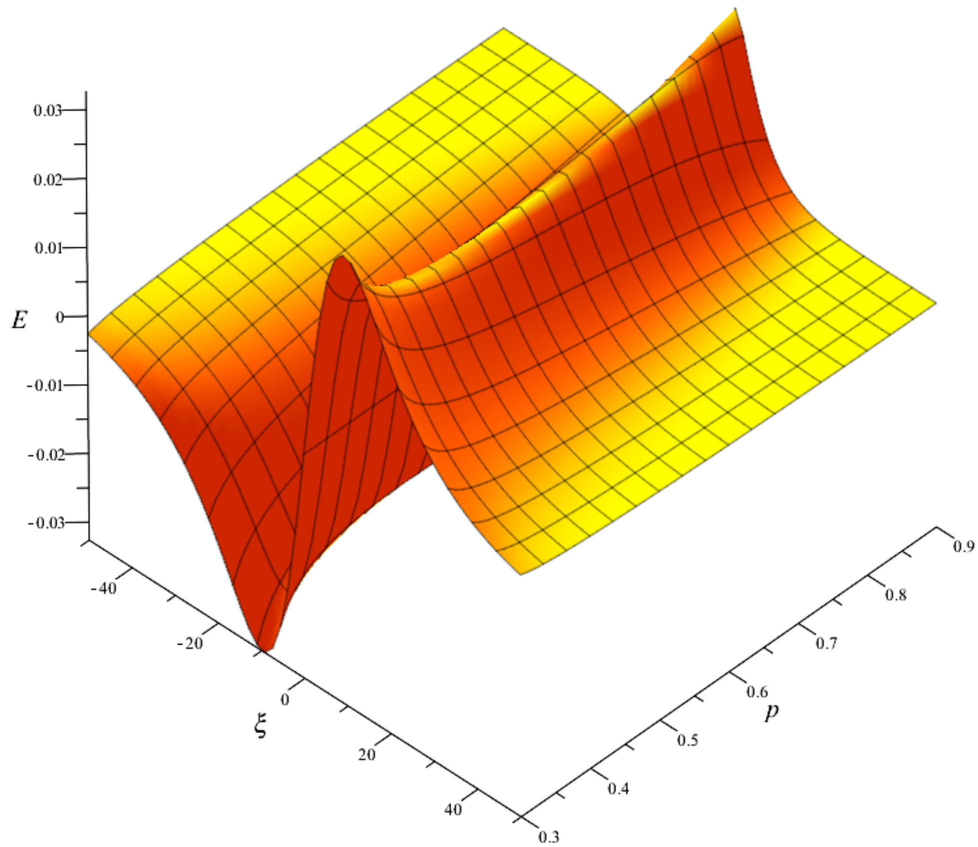

Figure 5.6: The normalized electric field with regards to $p > p_{SC}$ and ξ with $\beta = 0.7$, $\delta_{ie} = 0.1$, $\sigma = 1$, $U_0 = 0.0075$, $U_{i0} = 0.9 \times 10^8$ and $\beta_{ic} = 0.3$.

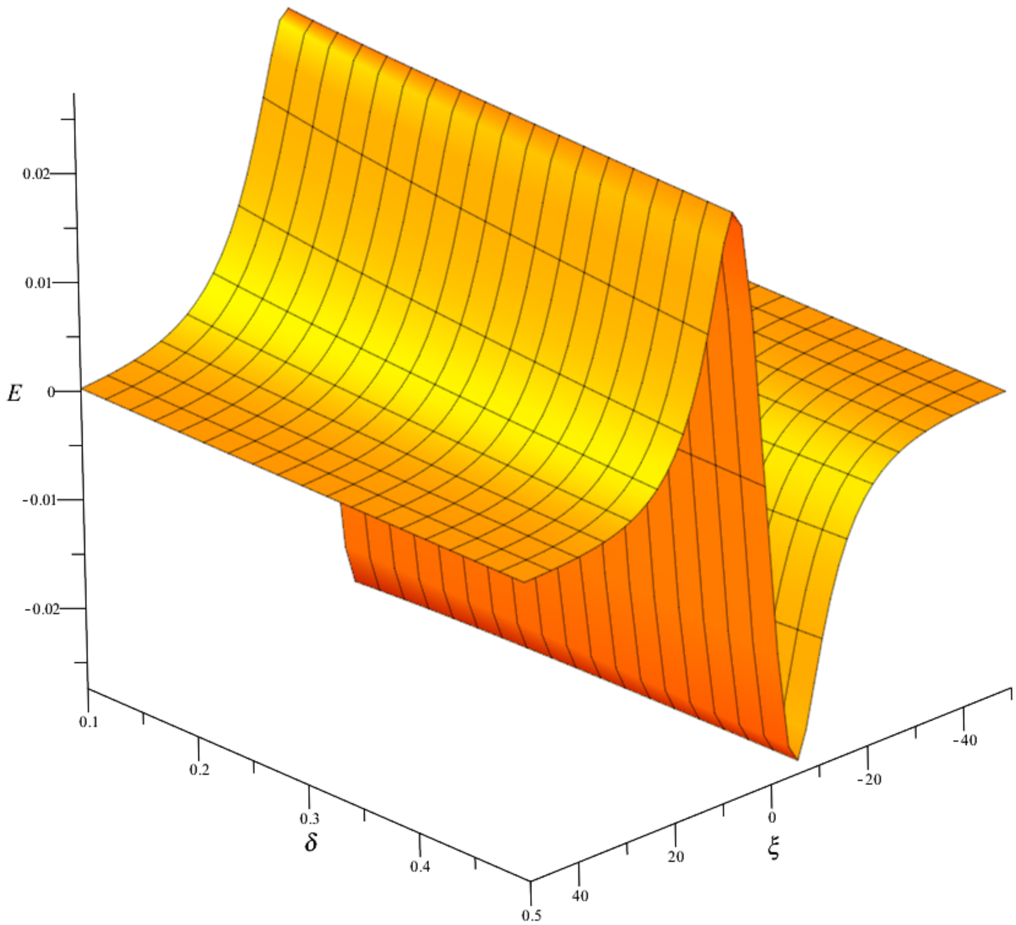

Figure 5.7: The normalized electric field with regards to δ_{ie} and ξ with $\beta = 0.5$, $p > p_{SC} = 0.3$, $\sigma = 1$, $U_0 = 0.0075$, $U_{i0} = 1.5 \times 10^8$ and $\beta_{ic} = 0.5$.

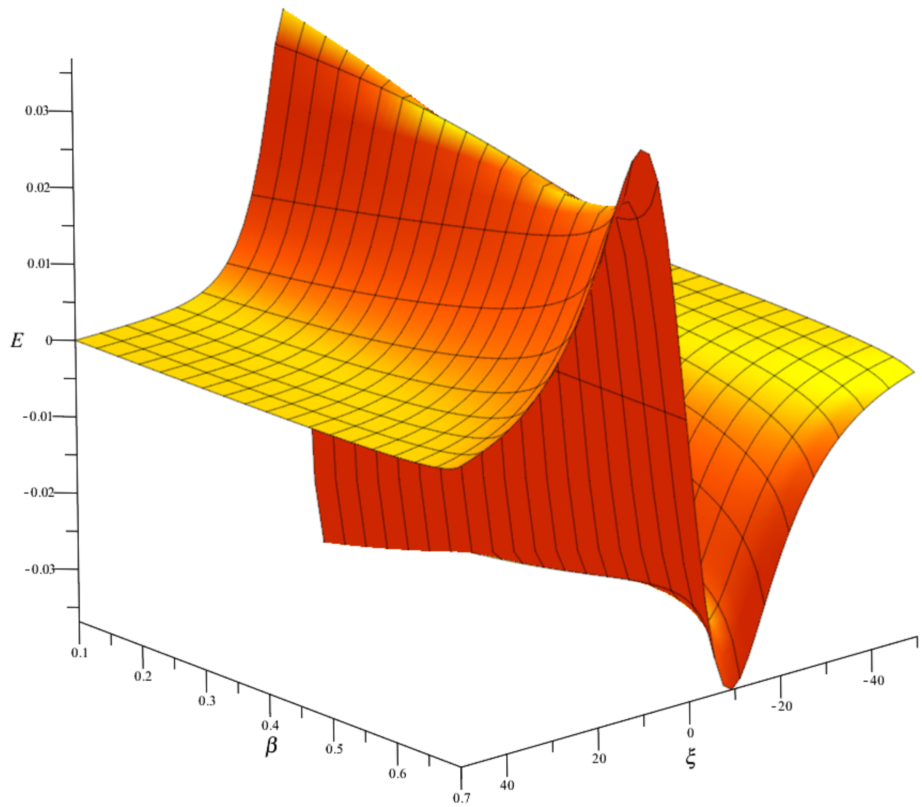

Figure 5.8: The normalized electric field with regards to $\beta_e = \beta_p = \beta$ and ξ with $p > p_{SC} = 0.3$, $\delta_{ie} = 0.1$, $\sigma = 1$, $U_0 = 0.0075$, $U_{i0} = 1.5 \times 10^8$ and $\beta_{ic} = 0.5$..

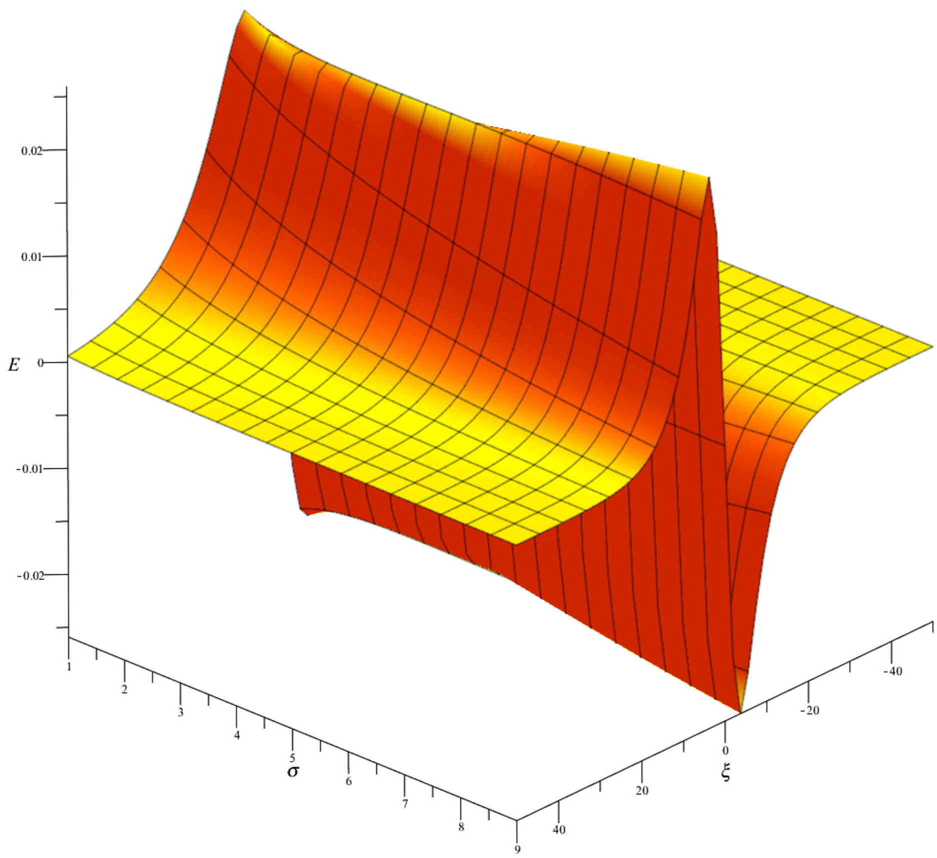

Figure 5.9: The normalized electric field with regards to σ and ξ with $p = 0.2$, $\delta_{ie} = 0.1$, $\beta = 0.5$, $U_0 = 0.0075$, $U_{i0} = 0.9 \times 10^8$ and $\beta_{ic} = 0.3$.

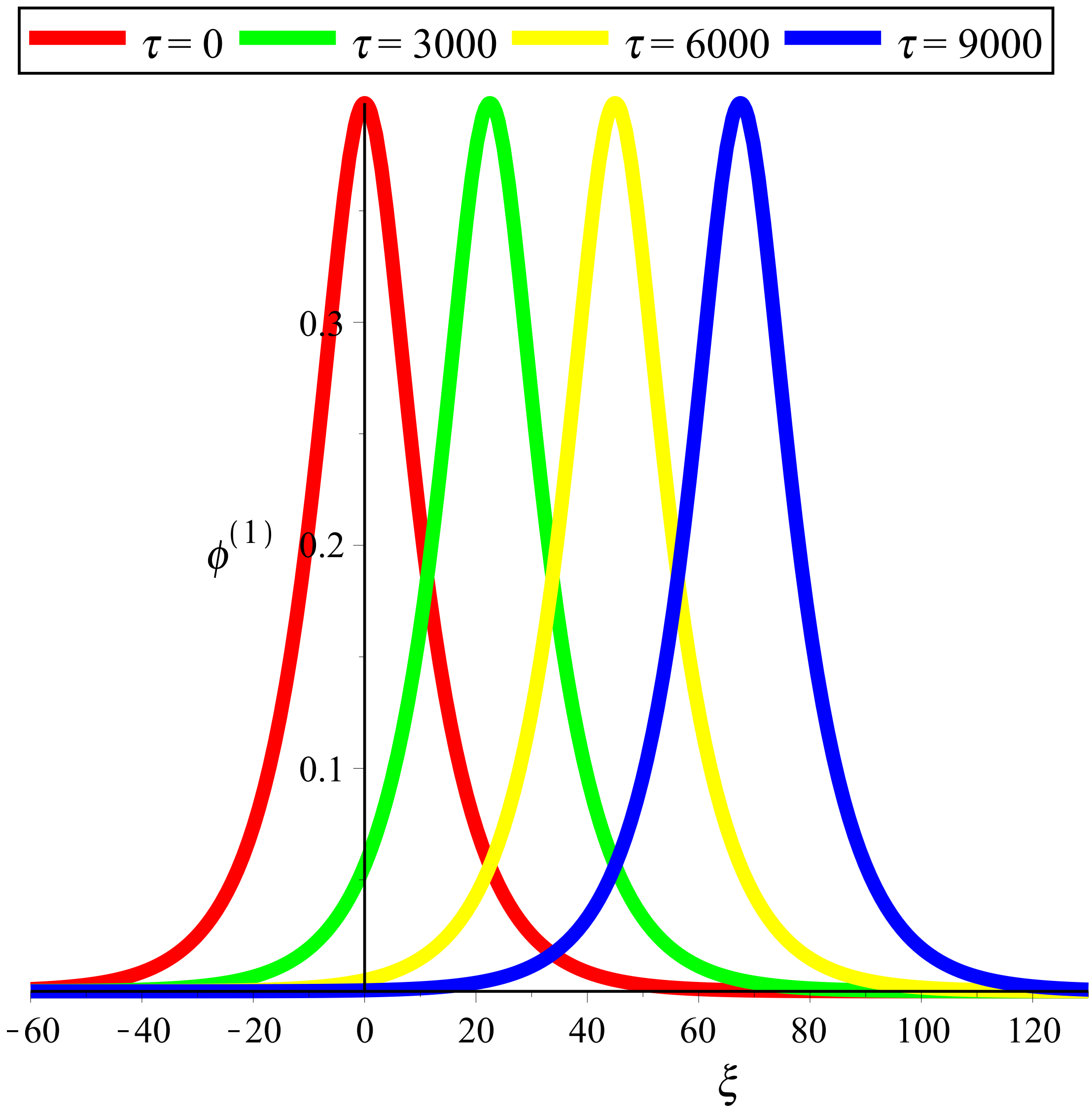

Figure 5.10: Variation of electrostatic potential with regards to time. The parametric values of the parameters are considered as $\beta = 0.3$, $\delta_{ie} = 0.1$, $\sigma = 1$, $U_0 = 0.0075$, $U_{i0} = 0.9 \times 10^8$ and $\beta_e = \beta_p = 0.5$ and $p > p_{SC} = 0.3$.

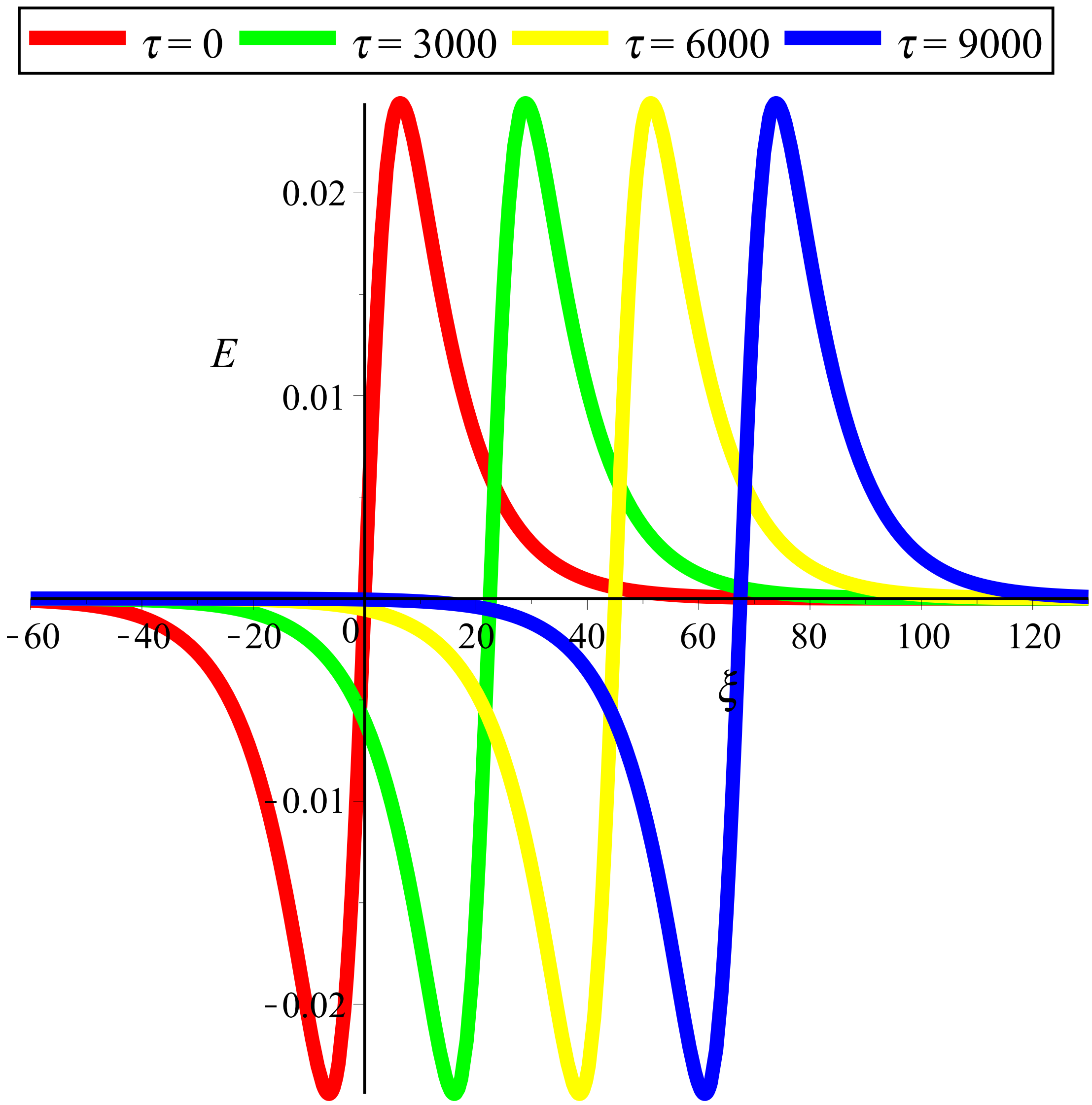

Figure 5.11: Variation of normalized electric field with regards to time. The parametric values of the parameters are considered as $\beta = 0.3$, $\delta_{ie} = 0.1$, $\sigma = 1$, $U_0 = 0.0075$, $U_{i0} = 0.9 \times 10^8$ and $\beta_e = \beta_p = 0.5$ and $p > p_{SC} = 0.3$.

5.4 CONCLUSIONS

The investigation of this chapter is an extension of the work made chapter 3 and 4 to analyze the nonlinear IASs with the appearance of the SCV. In this chapter, an unmagnetized plasma having relativistic ion fluids with nonthermal electrons and positrons has been studied to investigate the propagation of IASs around the SCV in the strongly relativistic regime by taking the RLF up to 20 terms. To accomplish the objectives, the quartic KdVE has been derived by applying the conventional reductive perturbation method, with the needed adjustments to the stretching coordinates for the first time. It is found that the considered plasma environment supports the SCVs by setting the nonlinear coefficient of the KdV and mKdV equal to zero. The impacts of plasma parameters on the properties of electrostatic IASs and their corresponding electric fields have been investigated by the soliton solutions of the quartic KdVE. It is observed that in this plasma environment, the compressive electrostatic soliton propagation exists around the SCV. The electrostatic IASs are formed bell-shaped type structures, whereas the corresponding normalized electric field is formed semi-kink shaped type structures. It is also found that the amplitudes and widths of IASs increase (decrease) with the increase of relativistic streaming factor (density and temperature ratio's) around the SCV. It may be concluded that the investigations made in this article would be helpful to understand the propagation characteristics of electrostatic IASs around SCVs not only in plasma sheath boundary layer of earth magnetosphere [28], Laser-plasma interaction [19, 29], quark-gluon [30], interstellar medium [3, 23], etc. but also in plasma laboratory [22].

Chapter 6: CONCLUDING REMARKS AND FUTURE ASPECTS

This chapter summarizes the findings from preceding chapters focusing on fully ionized, collisionless three-component unmagnetized relativistic plasma systems. Various plasma assumptions were considered, exploring nonlinear dynamics of IAWs through nonlinear evolution equations. The reductive perturbation method was utilized to analyze electrostatic solitary waves and solitons within these plasma contexts.

Chapter 1 introduced fundamental concepts and the presence of three-component unmagnetized relativistic plasma in astrophysical, space, and laboratory settings. Also, Plasma nonlinearity, wave phenomena, IAws, IASs and the statement of purpose have been discussed in the this chapter.

Chapter 2 deals with the fluid description, model equation, various types of nonlinear evolution equation and the methodology to evaluate the Nonlinear equations.

In Chapter 3, A relativistic unmagnetized plasma composed of nonthermal electrons, nonthermal positrons, and relativistic ion fluids was studied to report the propagation of IASs. By applying the reductive perturbation technique (RPT), the KdV equation was derived and its solution provided. The study examined how plasma parameters affect the propagation characteristics of IASs, modeled by the KdVE with up to 20 terms of the Lorentz relativistic factor (LRF). It was found that in the proposed relativistic plasma environment, both compressive and rarefactive IASs are supported in the presence of nonthermality. The amplitudes and widths of IASs are significantly influenced by increasing the values of the plasma parameters. Additionally, the relativistic streaming factor notably alters the nonlinear propagation of IASs, with soliton energy showing a slight increase when the relativistic streaming index exceeds 0.1 and a considerable increase when it remains below 0.1. In Chapter 4, we extend the research from Chapter 3 by examining a weakly relativistic unmagnetized plasma composed of nonthermal electrons and positrons, and weakly relativistic ion fluid. Here, the mKdVE is derived using the RPT method, incorporating coordinate adjustments in the presence of critical values (CVs). The impacts of plasma parameters on the properties of electrostatic IASs and their corresponding electric fields have been investigated using the soliton solutions of the mKdVE. The findings indicate that (i) the considered plasma environment supports critical values (CVs) and (ii) compressive electrostatic soliton propagation exists around these CVs. The electrostatic IASs form bell-shaped structures, while the corresponding normalized electric fields form semi-kink shaped structures. Additionally, it was observed that the amplitudes and widths of IASs increase with the rise of the relativistic streaming factor and decrease with the increase of the density and temperature ratios.

Chapter 5 delved into highly relativistic regimes, deriving quartic KdVE to study IASs in an unmagnetized relativistic plasma, extending the work of previous chapters by deriving the quartic KDVE around the SCV. In this chapter, an unmagnetized plasma having relativistic ion fluids with nonthermal electrons and positrons has been studied to investigate the propagation of IASs around the SCV in the strongly relativistic regime by taking the RLF up to 20 terms. It is found that the considered plasma environment supports the SCVs by setting the nonlinear coefficient of the KdV and mKdV equal to zero. The impacts of plasma parameters on the properties of electrostatic IASs and their corresponding electric fields have been investigated by the soliton solutions of the quartic KdVE. It is observed that in this plasma environment, the compressive electrostatic soliton propagation exists around the SCV. The electrostatic IASs are formed bell-shaped type structures, whereas the corresponding normalized electric field is formed semi-kink shaped type structures. It is also found that the amplitudes and widths of IASs increase (decrease) with the increase of relativistic streaming factor (density and temperature ratio's) around the SCV.

Overall, the chapters collectively underscored the significant role of ion streaming factors and other plasma parameters in shaping the dynamics of IA waves across different relativistic regimes, offering insights applicable to various astrophysical, space, and laboratory plasma environments.

Further investigation is needed to fully understand the specific influences of weakly and highly relativistic effects on the formation and dynamics of not only solitary but also shock structures in plasma systems. Additionally, future research could benefit from numerical simulations to validate theoretical findings and explore more complex plasma scenarios beyond the limitations of analytical methods. Moreover, theoretical predictions should be experimentally tested in laboratory plasma environments to confirm their applicability to real-world astrophysical and space plasma phenomena.

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