ISOTROPIC SOLUTION OF EINSTEIN'S EQUATIONS ON THE FLUID SPHERE

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Declaration

I hereby affirm that the work presented in this Thesis has not been previously submitted to fulfill the requirements for an award at this or any other institution of higher education. To the best of my knowledge and belief, the Thesis contains no material previously published or authored by another individual, preserve where appropriate citations are provided. Moreover, the Thesis adheres to the PLAGIARISM and ACADEMIC INTEGRITY regulations of CUET.

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Dedication

My All Family Members, My Friends

Respected Teachers

and

Department of Mathematics, CUET.

List of Publications

[1] M. I. H. Sakib et al., “Generating static spherically symmetric solution of Einstein’s equations in general relativity," *Abstract Published in International Conference on Physics for Sustainable Development and Technology (ICPSDT 2023).*

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Approval/Declaration by the Supervisor(s)

This is to certify that Md.Imran Hossain Sakib has carried out this research work under our supervision, and that he has fulfilled the relevant Academic ordinance of the Chittagong University of Engineering & Technology, so that he is qualified to submit the following thesis in the application for the degree of MASTER OF PHILOSOPHY IN MATHEMATICS. Furthermore, the Thesis complies with PLAGIARISM and ACADEMIC INTEGRITY regulation of CUET.

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Abstract

This thesis investigates the theoretical framework of spherically symmetric perfect fluid spheres within the realm of general relativity. Our work seeks exact solutions to the Einstein field equations thereby facilitating a better knowledge of minuscule objects. The thesis begins by examining the fundamental aspects of special relativity, including the Lorentz transformations and the structure of spacetime. We then explore general relativity, with an eye towards the Einstein field equations and the geometric justification of gravity. Examining matter dispersion in astrophysical systems depends on first looking at the behavior of ideal fluids.

The thesis delves into the intricacies of the Einstein field equations and their application to spacetimes with spherical symmetry. The key achievement of the research is the formulation of new exact solutions for static spherically symmetric ideal fluid spheres. Implementing two techniques in the context of static spherically symmetric ideal fluid line elements, we develop two new approaches that validate their characteristics and ensure that they satisfy Buchdhal criteria.

The physical characteristics and possible astrophysical implications of these solutions are thoroughly investigated. Combining the ideas of general relativity with the simplified model of perfect fluids helps this work provide new understanding on the interactions between compact objects and their gravitational interactions.

<u>বিমূর্ত</u>

এই গবেষণাকাজটি সাধারণ আপেক্ষিকতার কাঠামোর মধ্যে গোলাকার-প্রতিসম সম্পন্ন নিখুঁত তরল গোলকের তাত্ত্বিক কাঠামো অনুসন্ধান করে। আমাদের কাজটি আইনস্টাইনের ক্ষেত্র সমীকরণের সঠিক সমাধান খুঁজে বের করার চেষ্টা করে, যা ক্ষুদ্রাকৃতির বস্তুগুলির সম্পর্কে উন্নত ধারণা প্রদান করে। গবেষণার শুরুতে আমরা বিশেষ আপেক্ষিকতার মৌলিক বিষয়গুলি, যেমন লরেঞ্জ রূপান্তর এবং স্থান-কালের গঠন আলোচনা করেছি। এর পর সাধারণ আপেক্ষিকতা সম্পর্কে আলোচনা করা হয়েছে, যেখানে আইনস্টাইন ক্ষেত্র সমীকরণ এবং মহাকর্ষের জ্যামিতিক ব্যাখ্যার উপর গুরুত্ব দেওয়া হয়েছে।

জ্যোতির্বৈজ্ঞানিক সিস্টেমে পদার্থের বিস্তরণ বিশ্লেষণ করতে আদর্শ তরলের আচরণের দিকে মনোযোগ দেওয়া প্রয়োজন। এই গবেষণাকাজে আইনস্টাইনের ক্ষেত্র সমীকরণ এবং গোলাকার-সমমিতি সহ মহাকাশ-সময়ের ক্ষেত্রে এগুলোর প্রয়োগের জটিল দিকগুলো নিয়ে আলোচনা করা হয়েছে। গবেষণার প্রধান সাফল্য হলো, স্থির গোলাকার-সমমিতি সম্পন্ন আদর্শ তরল গোলকের জন্য নতুন কিছু সঠিক সমাধান তৈরি করা।

স্থির গোলাকার-সমমিতি আদর্শ তরল রেখাংশের প্রেক্ষাপটে দুটি কৌশল প্রয়োগ করে, আমরা দুটি নতুন পদ্ধতির উন্নয়ন করেছি যা তাদের বৈশিষ্ট্য যাচাই করে এবং এগুলো বুশডাল মানদণ্ড পূরণ করে কিনা তা নিশ্চিত করে। এই সমাধানগুলির ভৌত বৈশিষ্ট্য এবং সম্ভাব্য জ্যোতির্বৈজ্ঞানিক প্রভাবগুলো বিশদভাবে বিশ্লেষণ করা হয়েছে। সাধারণ আপেক্ষিকতার ধারণার সাথে আদর্শ তরলের সরলীকৃত মডেল একত্রিত করে এই কাজটি কমপ্যাক্ট বস্তুর মধ্যে মহাকর্ষীয় মিথস্ক্রিয়া সম্পর্কে ধারণা প্রদান করতে সক্ষম হয়েছে।

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Nomenclature

MCRF	Momentarily Comoving Reference Frame
GR	General Relitivity
QM	Quantum Mechanics
GZK	Greisen-Zatsepin-Kuzmin
UHECRs	Ultra High Energy Cosmic Rays
$G_{\mu\nu}$	Einstein's Tensor
$T_{\mu\nu}$	Energy-Momentum Tensor
R	Ricci Scalar
$g_{\mu\nu}$	Metric Tensor
Λ	Cosmological Constant
$R_{\mu\nu}$	Ricci Tensor
$\nabla_{\mu}T^{\mu\nu} = 0$	Covariant Derivative
$\Gamma^{\lambda}_{\mu\nu}$	Christoffel Symbols
$R^{\rho}_{\sigma\mu\nu}$	Riemann Tensor
$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$	Lorentz Factor
$\otimes$	Tensor Product
$\Box$	D'Alembertian or wave operator

Chapter 1: INTRODUCTION

1.1 THE EXACT SOLUTIONS OF EINSTEIN'S FIELD EQUATION AND THE IMPORTANCE OF STUDY THEM

Theoretical frameworks in modern physics often rely on a certain family of differential equation systems. Applying these equations to a predefined set of laws allows us to derive helpful knowledge about the physical world from the mathematical outcomes. Of all the theories of gravitation, general relativity of Einstein's is usually regarded as the finest. Differential equations in this context are determined by the geometric characteristics of a Riemannian (Lorentzian) manifold, which stands for space and time. Apart from elucidating Einstein's well-known field equations, these equations clarify the connection between matter and gravity,

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = k_0 T_{ab} \tag{1.1}$$

First of great relevance is understanding and organizing Einstein's theory. This thesis's everything is based on Einstein's theory. While we won't delve into all the specifics, our ability to comprehend and interpret the theory of general relativity is going to impact its development. Development of any physical theory starts with mathematical analysis of the system of differential equations; the objective is to identify a general solution or multiple exact answers. After obtaining the answers, the next phase is to ascertain their mathematical and physical meanings. This reading transcends the local solution of differential equations inside the paradigm of general relativity. For this, one needs topological methods and global analysis. Gravity theories face still another set of difficulties in the effect of the gravitational field on other fields and materials. Usually an exact answer, physicists studying such issues do so within a specified gravitational field. This thesis primarily focuses on the solutions to the Einstein equation (1.1) for a stationary, spherically symmetric perfect fluid. It briefly touches on several other topics as well. Unfortunately, there has often been a weak connection between the study of exact solutions and the investigation of simpler physical conditions. As Kinnersley noted in 1975, many of the known exact solutions describe physically unrealistic situations, which can divert attention from the more practical ones. However, the issue is also partly due to those of us working in this field. In our pursuit of greater generality, we sometimes introduce null currents, macroscopic neutrino fields, and tachyons, enjoying the creation of complex and unintuitive notation. As a result, we often leave our newly derived metrics without a clear interpretation. Nevertheless, there are times when we may encounter measures that are not easily understood, raising more questions than answers. Despite the challenges in this field, much has remained unchanged since then. Our efforts remain valuable and essential for advancing the discipline.

Given the significance of specific answers in these discussions, it is abundantly evident that solving physical challenges requires working on precise solutions. Black holes are characterized by the Schwarzschild and Kerr solutions; the Friedmann solutions apply to cosmology; and the plane wave solutions helped to settle disputes on gravitational radiation. It should be mentioned that the very non-linear character of general relativity often makes it difficult to grasp the qualitative features of solutions. A great resource has been the precise answers, some of which could seem odd. Ehlers et al. [5] pointed out that non-linearities in general relativity might cause unexpected problems for perturbation methods. The availability of exact solutions that may be compared with numerical or approximative results considerably strengthens the validity of approximation techniques. Amazingly, there are already quite many known exact answers. The whole Einstein equations are rather complex; hence many non-specialists mistakenly believe that there can only be so many solutions. From one point of view, this is accurate: not yet do we know all the precise solutions to physical problems that surface in the real world. Still, given many unresolved issues, there is plenty of space for creative developments in this field. These address our non-uniform universe, the two-body problem, looking at the gravitational field of a stationary rotating star, and how gravitational radiation is produced and spread from a confined source. Regarding other matters, it is probable that just the precise established solutions would be advantageous. An illustration of this can be seen in the Kerr and Schwarzschild solutions, which describe the final state of massive objects after they have collapsed. In this field, there is great expectation for creative ideas and major developments in the further future. Furthermore, we can proceed with applying metric symmetry constraints, limiting the algebraic structure of the Riemann tensor, including field equations for the matter variables, or defining starting and boundary conditions. We have found all the known exact solutions by restricting the problem. An exact solution is a metric specified in terms of well-known analytic functions in the proper coordinates, such polynomials, trigonometric functions, hyperbolic functions, etc. Eliminating functions defined by (linear) differential equations by themselves presents a challenging task. Though it suggests a thorough awareness of the characteristics of the metric, the word "exact solution" does not offer the degree of precision that would be ideal. Still, none of the people engaged have agreed on the particular meaning of this term. Actually, we chose to include a perfect answer; so, we proceeded with that instead. The initial stages of general relativity research yielded limited and conclusive findings. These objects exhibited a high degree of symmetry and were primarily influenced by idealized physical considerations. The axisymmetric static electromagnetic and vacuum solutions of Weyl, along with the plane wave metrics and spherically symmetric solutions such as those of Schwarzschild, Reissner, Nordström, Tolman, Friedmann, Robertson, Walker, and others, are discussed.

Although the suggested solutions were very limited, it should be mentioned that they almost completely cover the important ones in practical applications. The Kerr solution, discovered following World War II and equally significant as these previous solutions, explains spacetime around a spinning mass; this is a typical occurrence in the cosmos. Early days, when few people were interested in general relativity, other than maybe as models for stars and cosmology, it was generally believed that perfect solutions would be of limited use. This point of view was maintained as it was believed that the relativistic adjustments to Newtonian gravity had minimal effect. Emphasizing approximations, particularly the weak-field, slow-motion approximation, several physical issues were addressed at this period. This historical backdrop helps one to see the development and growth of general relativity. For generating solutions, there are a set of mathematical tools and algorithms available that create new solutions from existing ones. These techniques were still in their early years of evolution at the time. The approaches of the area have produced many new ideas, therefore highlighting the fascinating and continuous character of general relativity research.

1.2 THE OUTLINE OF THE THESIS

chapter 2- This chapter has covered the basics of special relativity, including its mathematical structure and key ideas. The theory's foundation, the Lorentz transformations, has been extensively described and shown to fully understand the invariance of the speed of light within the principles of relativity. Time dilation, relativistic velocity addition, and length contraction are all phenomena that have been explained using this paradigm. These paradoxes have been proven by science. The integration of time and space into the Minkowski spacetime manifold provides a geometric viewpoint on special relativity kinematics. When considering the inclusion of gravity, the limitations of this theory become apparent, leading to the development of general relativity.

chapter 3- General relativity is introduced fundamentally in this chapter. The idea effectively combined acceleration and gravity beginning with the equivalency principle. The metric tensor allowed us to formally propose and define the idea of gravity as a curvature of spacetime. Emphasizing the generic covariance principle, one may show how the theory is indifferent to specific coordinate systems. Essential to the theory, the Einstein field equations first emerged as the dynamic equations controlling spacetime. By means of these equations, the Schwarzschild solution which provides a useful foundation for comprehending gravitational fields around large objects was discovered from spherically symmetric systems. At last, we explored the connection between general and special relativity, stressing the importance of the Minkowski metric as a particular case within the larger background.

chapter 4-The foundational concepts of perfect fluids in the framework of general relativity have been presented in this chapter. Now introducing the stress-energy tensor, an essential quantity defining momentum and energy distribution. A fundamental principle is investigating the conservation of energy-momentum in the framework of perfect fluids. In cosmological and astrophysical settings, idealized models of matter dispersion using perfect fluids have been highlighted. These ideas are fundamental for understanding the interaction between matter and geometry in the gravitational field as stated by Einstein's field equations.

chapter 5- Introduced and briefly discussed in this chapter are Newtonian gravity and the Einstein field equations of general relativity, which are bridged between by the weak-field limit and four-velocity. As this chapter shows, Einstein field equations can clarify the dynamical description of the matter-geometry interaction. While the solutions to these equations may be intricate, it is essential to understand the structure and evolution of small objects. Theoretical physics experts assert that the field equations offer a comprehensive and complex explanation of the relationship between spacetime geometry and gravity.

chapter 6- This chapter first addresses the characteristics of spherically symmetric perfect fluid spheres begin with an understanding of flat space in terms of spherical coordinates. The subject then turned to curved spacetime, with a particular focus to how it influences two-space and generates spherically symmetric spacetime structures. Research on static spherically symmetric spacetimes has much improved our understanding of equilibrium configurations of perfect fluid spheres. Ultimately, the importance of knowing the physical definition of metric concepts was highlighted so clarifying how these terms define the geometry and dynamics of these systems in the framework of general relativity.

chapter 7- This chapter investigates advanced solution generating methods for the Einstein field equations and generates the Static Spherically Symmetric Solutions for Einstein's Equation. We present two new solutions with particular characteristics by using two approaches, including those fulfilling the Buchdahl condition, therefore restricting the maximum compactness of a stable star.

chapter 8-These chapters taken together offer a thorough investigation of the theoretical foundations of relativity and their applications to perfect fluid spheres, therefore advancing our knowledge of gravitation, spacetime, and the behavior of matter. This chapter also highlight the significance of the two new solution from the known solution, analyzed there physical characteristics and verify the solutions within Buchdahl conditions, which play a important role in astrophysical aspects.

Chapter 2: SPECIAL RELATIVITY: FUNDAMENTAL INSIGHTS

2.1 INTRODUCTION

Modern physics principal framework, special relativity, offers a significant departure from conventional Newtonian mechanics. The fundamental ideas, mathematical formalism, and main consequences of this chapter are briefly summarized. In this chapter, we will look into the postulates of relativity, the Lorentz transformations, and their effects on the perception of space and time including time dilation, length contraction, and velocity addition. Minkowski spacetime will be introduced to bring time and space into one entity. Ultimately, the chapter will briefly discuss possible constraints of special relativity, therefore preparing the ground for an in-depth investigation of general relativity.

2.2 FOUNDATION OF SPECIAL RELATIVITY

Maxwell's significant accomplishment was the understanding of light as an occurrence of electromagnetic waves. In order to align with the widely accepted mechanical perspective of nature, it was imperative to postulate the presence of a medium called "the ether" that would serve as a conduit for these waves. Nonetheless, the discrepancies between classical mechanics and the equations governing electromagnetism, coupled with the results of the Michelson-Morley experiment which failed to identify the luminiferous ether, offered compelling evidence against the existence of the previously proposed ether.

The theory of composition was later developed by Einstein in his doctoral dissertation on relativity entitled, on the electrodynamics of moving bodies in 1905. Now, it is appropriately recognized by his name as a rational resolution to what physics previously perceived. Galileo developed the notion of relativity in physics by claiming that physical principles apply to all inanimate systems. This relativity principle implies the existence of absolute space in relation to which objects move. The concept of inertial frames refers to the limiting limits of reference frames in which objects are in uniform motion. From the beginning, an idea of the absolutism of time has been developed.

Einstein established an essential differentiation between inertial and non-inertial reference frames. This distinction highlights the significance of viewpoint in the realm of physical sciences. Although Maxwell's equations maintain their form under Lorentz transformations, they do not display the same consistency when applied to Galilean transformations. This is the reason behind the theory that the speed of light c, akin to all other fundamental principles, remains invariant across all inertial systems. The Michelson-Morley experiment has provided an indisputable contribution to the credibility of this specific theory of relativity, and the velocity of light has played a significant role in its establishment. Poincaré had previously broached the same concept, although it was Einstein who initially comprehended its complete significance and effectively used it [1,6–10].

2.3 PRINCIPLES OF SPECIAL RELATIVITY

Einstein thought that the idea of absolute motion was worthless. Einstein gave the Galileo principle (Galilean relativity) a central role in his special theory of relativity as the first postulate. The Michelson-Morley experiment and its subsequent follow up experiments established that the speed of light in a vacuum is constant, and unaffected by changes to point towards which it propagates.

This is also one of the assumptions in Einstein's second postulate. Both postulates are present

"In all Galilean frames, all physical laws remain invariant."

“The speed of light in a vacuum is the same for all observers, regardless of the motion of the light source or observer.”

The essence of Einstein's relativity principle is to apply it in physics (not only mechanics) as an extension of Newton's special theory, a high-dimensional generalization. It simply says that any physical measurements cannot determine whether an inertial system is at a fixed position or moving indeterminately, we can only speak relatively about motion of two systems. Therefore, no experiment performed entirely within the walls of an inertial frame can anywhere inform the experimenter as to how his system is moving with respect to any other inertial frame.

According to Einstein, Maxwell's equations are not different in different inertial frames of reference. Since they are not invariant for Galilean transformation, Einstein argued that the equations of transformation by Galilei transformations cannot lead to a suitable form of transition between different inertial frames. One year before Einstein formulated the special theory of relativity, Lorentz developed transformation equations which keep Maxwell's equations constant. For Lorentz, however, this was regarded as only psychological, mathematical effect - illusory change has nothing to do with actual physical reality (he considered the size contraction also) Einstein separately deduced Lorentz transformation equations and endowed them with physical significance. And these two assumptions made by Einstein are the basis of all special relativity. These would be some qualities of that genius; simplicity, courage and generality [1, 6–10].

2.4 LORENTZ TRANSFORMATION

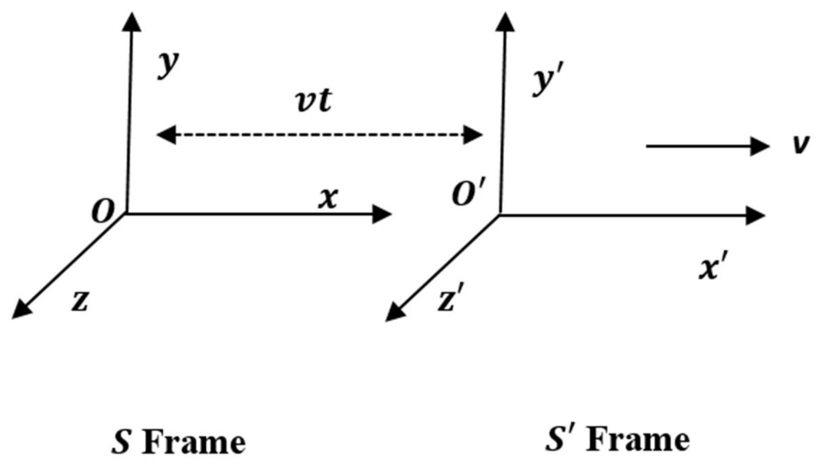

Figure 2.1: Lorentz Transformation Co-ordinates [1].

Examining a frame of reference that is inertial, characterized by coordinates x, y, and z at a specific time t. An occurrence takes place at time t' with coordinates x', y', and z', where υ is oriented along the positive x-axis, as perceived from an alternate inertial frame S', which is in motion relative to S with a constant velocity υ. One can now make a reasonable estimate regarding the appropriate relationship between x and x', allowing for an informed conjecture about the correct connection between these coordinates,

$$x' = k(x - \upsilon t) \tag{2.1}$$

Here,k is a constant that is independent of both x and t. Since physics equations need to have the same form in both S and S',we may write the appropriate expression for S in terms of x' and t' by simply changing the sign of v,

$$\left.\begin{aligned} x = k(x' + \upsilon t') \\ y = y' \\ z = z' \end{aligned}\right\} \tag{2.2}$$

Since there's no variation between S and S' other than the sign of v, the constant k should be the same in both frames of reference.

There is no indication that any discrepancies might exist between the corresponding coordinates y and y', or z and z', which are perpendicular to the direction of υ, as is the case with the Galilean transformation. Consequently, the time coordinates t and t' are not equivalent. By substituting the value of x' in equation (2.2), gives us,

$$\begin{aligned} x = \; & k^2(x - \upsilon t) + k\upsilon t' \\ \text{Or, } t' = \; & kt + \left(\frac{1-k^2}{k\upsilon}\right)x \end{aligned} \tag{2.3}$$

The second principle of relativity provides a method for calculating k. According to our initial conditions, the origins of the two frames of reference S and S' are at the same position at $t = 0$ as well as $t' = 0$ the same. Assume that at $t = t' = 0$ a light beam is launched from the common source of S and S', and the spectators in each region record the velocity with which the light beam travels outside. Both spectators must find the same velocity c, which implies that in the S and S' frame,

$$x = ct \tag{2.4}$$

$$x' = ct' \tag{2.5}$$

Putting the value of x' and t',

$$k(x-\upsilon t) = ckt + \left(\frac{1-k^2}{k\upsilon}\right)cx$$

$$\text{Or,}\;\; x = \frac{ckt+\upsilon kt}{k-\left(\frac{1-k^2}{k\upsilon}\right)c} \;=\; ct\left[\frac{k+\frac{\upsilon k}{c}}{k-\left(\frac{1-k^2}{k\upsilon}\right)c}\right]$$

$$\text{Or,}\;\; x = ct\left[\frac{1+\frac{\upsilon}{c}}{1-\left(\frac{1}{k^2}-c\right)\frac{c}{\upsilon}}\right] \tag{2.6}$$

This is the same equation for x.The above equation will be true if the quantity in the brackets is equal to 1. Therefore,

$$\frac{1+\frac{\upsilon}{c}}{1-\left(\frac{1}{k^2}-c\right)\frac{c}{\upsilon}} = 1$$

$$k = \frac{1}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$$

Finally, plugging this k value into equations (2.1),(2.2) and (2.3). We now have a complete transition of measurements taken during an event in S to measurements taken in S'.

$$\left.\begin{aligned} x' &= \frac{x-\upsilon t}{\sqrt{1-\frac{\upsilon^2}{c^2}}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t-\frac{\upsilon x}{c^2}}{\sqrt{1-\frac{\upsilon^2}{c^2}}} \end{aligned}\right\} \tag{2.7}$$

These equations comprise the Lorentz transformation [7]. Now the inverse Lorentz transformation equations are,

$$\left.\begin{aligned} x &= \frac{x'+\upsilon t}{\sqrt{1-\frac{\upsilon^2}{c^2}}} \\ y &= y' \\ z &= z' \\ t &= \frac{t'+\frac{\upsilon x}{c^2}}{\sqrt{1-\frac{\upsilon^2}{c^2}}} \end{aligned}\right\} \tag{2.8}$$

2.5 TIME DILATION

The velocity with which the observer is moving in relation to what they witness will impact how measured really are these time scales. In short, a clock runs much slower than an observer running at any appreciable fraction of the speed whereupon things should seem to slow down and occur less frequently when they take place in another inertial frame. For the moving spaceship observer who observes a time interval between two occurrences, we must see on Earth that these are happening over a long period of t. The best time for this interval between the two events is by an event happening at a single location in observer S'frame. This period makes distinct points when viewed from the ground-beginning and end marked events of the time interval happen at different places in spacetime, making it last longer than proper time. This is referred to as time expansion. So, we can say,

"Moving clock ticks slower than a stationary clock."

To demonstrate this, imagine a clock positioned at point x' within the moving frame S'. When an observer in S' notes the time as $t_1^{'}$, the corresponding time recorded in the frame S is t_1. Referring to equation (2.8), this relationship can be further analyzed,

$$t_1 = \frac{t_1^{'} + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

According to his clock, the observer in the moving frame observes that the time is now $t_2^{'}$, after a time, delay of t_0. So,

$$t_0 = t_2^{'} - t_1^{'}$$

However, the observer in S perceives the end of the same time interval as occurring at a different time,

$$t_2 = \frac{t_2^{'} + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$t = t_2 - t_1 = \frac{t_2^{'} - t_1^{'}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$t = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}} \tag{2.9}$$

$t_0 > t$, as shown in equation (2.8), indicates that time is dilated [7].

2.6 LENGTH CONTRACTION

Length contraction is a fascinating phenomenon in which the observed length of a moving object appears shorter than its proper length, which is measured when the object is stationary in its original inertial frame. This effect, known as Lorentz contraction or Lorentz-FitzGerald contraction, becomes significant only at speeds approaching a substantial fraction of the speed of light. Both spatial and temporal measurements are affected by relative motion, and it is well established that the length L of an object appears contracted when the object is moving relative to the observer. It is crucial to note that this contraction occurs only in the direction of the object's motion. The proper length of an object in its rest frame is denoted as L_0.

Consider a rod in the moving frame S' that is parallel to the x'-axis. The coordinates of the rod's ends are x'_1 and x'_2 as measured by an observer in this frame. The proper length of the rod is then given by

$$L_0 = {x'}_2 - {x'}_1$$

To find $L = x_2 - x_1$, the rod's length as measured in the stationary frame S at t, using equation (2.7),

$${x'}_1 = \frac{x_1 - \upsilon t}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}$$

$${x'}_2 = \frac{x_2 - \upsilon t}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}$$

So,

$$L = x_2 - x_1 = \frac{{x'}_2 - {x'}_1}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}$$

$$L = \frac{L_0}{\sqrt{1 - \frac{\upsilon^2}{c^2}}} \tag{2.10}$$

$L_0 > L$ demonstrates that the length is contracted in equation (2.9) [6].

2.7 VELOCITY ADDITION FORMULA

Assume that something moves in relation to both S' and S. The three velocity components of an observer in S' are measured to be,

$$V_x = \frac{dx}{dt}$$

$$V_y = \frac{dy}{dt}$$

$$V_z = \frac{dz}{dt}$$

To an observer in S' frame, however,

$$V'_x = \frac{dx'}{dt'}$$

$$V'_y = \frac{dy'}{dt'}$$

$$V'_z = \frac{dz'}{dt'}$$

Also, we obtain the inverse Lorentz transformation as:

$$\left.\begin{aligned} dx &= \frac{dx' + \upsilon dt'}{\sqrt{1-\frac{\upsilon^2}{c^2}}} \\ dy &= dy' \\ dz &= dz' \\ dt &= \frac{dt' + \frac{\upsilon dx'}{c^2}}{\sqrt{1-\frac{\upsilon^2}{c^2}}} \end{aligned}\right\} \tag{2.11}$$

And so,

$$V_x = \frac{dx}{dt} = \frac{dx' + \upsilon dt'}{dt' + \frac{\upsilon dx'}{c^2}}$$

$$V_x = \frac{\frac{dx'}{dt'} + \upsilon}{1 + \frac{\upsilon}{c^2}\frac{dx'}{dt'}}$$

So, the relativistic velocity transformation,

$$V_x = \frac{V_x' + \upsilon}{1 + \frac{\upsilon}{c^2} V_x'} \tag{2.12}$$

Similarly,

$$V_y = \frac{V_y' \sqrt{1 - \frac{\upsilon^2}{c^2}}}{1 + \frac{\upsilon}{c^2} V_x'} \tag{2.13}$$

$$V_z = \frac{V_z' \sqrt{1 - \frac{\upsilon^2}{c^2}}}{1 + \frac{\upsilon}{c^2} V_x'} \tag{2.14}$$

If $V_x' = c$ the same speed of light will be measured by the observer in both frames [7].

2.8 MINKOWSKI SPACE-TIME M_4

Lorentz transformation inspires us to consider time t as a part of four-dimensional space-time, called the Minkowski space-time M_4. Any point of M_4 is described by four co-ordinates (t, x, y, z) called an event, where t indicates the time when the event occurs and (x, y, z) indicates the place where it occurs. Thus, the Minkowski space-time M_4 is composed of all events. It can be shown that M_4 is a 4-dimensional manifold. The crucial thing which makes M_4 a Riemannian is the fact that the interval between two events is invariant under transformation between inertial frames. Suppose two nearby events have co-ordinates (t, x, y, z) and (t+dt, x+dx, y+dy, z+dz) relative to some inertial frames S. Then, the interval between two events ds^2 is defined by,

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \tag{2.15}$$

The time duration between two events remains constant regardless of Lorentz transformation. This implies that if the events corresponding to S', then the events have coordinates (t', x', y', z') and (t'+dt', x'+dx', y'+dy', z'+dz') respectively. Now from inverse Lorentz transformation equations (2.11) and let, $\gamma = \frac{1}{\sqrt{1- \frac{\upsilon^2}{c^2}}}$, then we can simply prove that $ds^2 = ds'^2$ is invariant. So, we can write,

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

$$= c^2\gamma^2\left(dt' + \frac{\upsilon dx'}{c^2}\right)^2 - \gamma^2(dx' + \upsilon dt')^2 - dy'^2 - dz'^2$$

$$= c^2\gamma^2\left\{(dt')^2 + 2.dt'.\frac{\upsilon dx'}{c^2} + \left(\frac{\upsilon dx'}{c^2}\right)^2\right\} - \gamma^2\left\{(dx')^2 + 2.dx'.\upsilon dt' + (\upsilon dt')^2\right\}$$
$$- dy'^2 - dz'^2$$

$$= c^2\gamma^2(dt')^2\left(1 - \frac{\upsilon^2}{c^2}\right) - \gamma^2\,(dx')^2\left(1 - \frac{\upsilon^2}{c^2}\right) - dy'^2 - dz'^2$$

$$= c^2dt'^2 - dx'^2 - dy'^2 - dz'^2$$

$$= ds'^2.$$

Hence (2.15) can serve as a metric on the Minkowski space-time M_4. Minkowski space-time is a Riemannian manifold because it has a matric. Following tensorial notations let us put x^0=ct, x^1=x, x^2=y, x^3=z, then we get,

$$ds^2 = d(ct)^2 - dx^2 - dy^2 - dz^2$$

$$= (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

$$= \eta_{\mu\nu}dx^\mu dx^\nu$$

so, $\mu, \nu = 0,\ 1,\ 2,\ 3$, Where, $\eta = \ diagonal(1,\ -1,\ -1,\ -1)$ In terms of S' inertial frame we have,

$$ds^2 = \eta_{\mu\nu}dx'^\mu dx'^\nu$$

This means that the metric,

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu \tag{2.16}$$

is form-invariant under transformations between inertial frames. Now, if we transform to a non-inertial frame with co-ordinates y^{μ} then the transformation equations,

$$\left.\begin{aligned} y^0 &= y^0(x^{\mu}) \\ y^1 &= y^1(x^{\mu}) \\ y^2 &= y^2(x^{\mu}) \\ y^3 &= y^3(x^{\mu}) \end{aligned}\right\} \tag{2.17}$$

are no longer Lorentz transformations. In fact, equations (2.16) proves to be non-linear equations. In terms of co-ordinate y^{μ} the metric (2.16) reduces to the general form,

$$ds^2 = g_{\mu\nu} dy^{\mu} dy^{\nu}$$

where,

$$g_{\mu\nu} = \frac{\partial x^{\sigma}}{\partial y^{\mu}} \frac{\partial x^{\lambda}}{\partial y^{\upsilon}} \eta_{\sigma\lambda}$$

The event (t, x, y, z) will henceforth be represented by a point with these co-ordinates in 4-dimensional space i.e Minkowski-space. Consider that, $t = 0$ a particle is at the origin O of S and it starts moving along x-axis with uniform speed u. So, its y or z will always remain zero on space-time representation we are concerned about position as function of time only, so this motion will be shown in x-t plain. In this plain, its motion will appear as the straight line QP, Q being the point x = y = z = t = 0 (figure-2.2). QP is called the world-line of the particle. Let $\angle PQt = \theta,\ tan\theta = u.\ But\ |u| \leq c$ and hence the world line of the particle must lie in the sector AQB, where $\angle AQB =\ 2\alpha\ and\ tan\alpha = c$. Similarly, for a particle following the Ox axis and passing through the point O at $t = 0$ its trajectory must lie in zone $A'QB'$. As the particle can be at both points simultaneously, it follows that every event in any of these domains must have had occurred with a time like interval from Q. Events in the sector AQB', $A'QB$ are separated from Q by space-like intervals. On the lines AA' and BB', $x = \pm ct$. Now turn our attention to those events whose world point is in the AQB region. Here, $t > 0$ in this region, i.e. all the events in this region occurs "after" Q but two events separated by a time-like interval cannot happen at the same time in any reference system. This implies that the events of region AQB can never have taken place

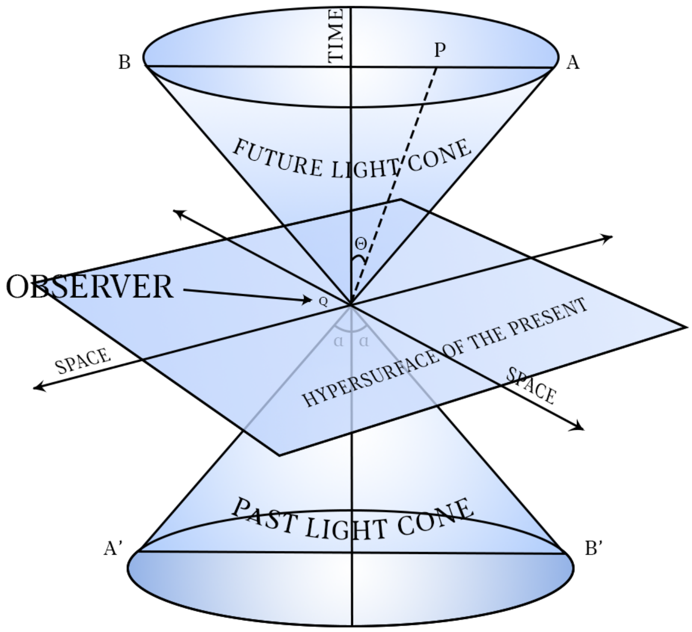

Figure 2.2: Light cone. No events can be causally related if one of them lies outside the cone [2].

"before" event Q, i.e., they all lie at times $t < 0$ in any reference system. Therefore, all the events from region AQB is future and therefore in general can be called an absolute future with respect to Q. Equally well, all of them belong to $A'QB'$ are absolutely past; that events in this region happen earlier than the event Q in any systems.

Next consider regions AQB' *and* $A'QB$. The interval between any two events in this region and event Q is space-like. These events all take place at different places in space with respect to every reference frame. Thus, these areas are referred to as possibly present. If we include the space co-ordinates so for 3 Space co-ordinates, instead of two lines that intersect at a point in figure-2.2 We have cone.

$$x^2 + y^2 + z^2 - c^2t^2 = 0$$

The time axis on the cone coincided with t, x, y and z axes (four-dimensional co-ordinate system) x, y, z, t. This cone is nothing but the light-cone.

2.9 SPACETIME WHEEL

A spacetime wheel representation is shown in figure-2.3. In this graphic representation, we can observe the concept of spacetime in which x-axis represents space and y-time. So, a Lorentz boost is nothing but taking place of an x rotation in the spacetime wheel between time t and space x, during which invariant hyperbola describes one point on our rotating object. A point on the wheel has changing spacetime coordinates $\{t, x\}$ with respect to its centre, but the spacetime spacing (s) between the point and the centre stays constant,

$$s^2 = -t^2 + x^2 = \; constant \tag{2.18}$$

More generally, the spacetime coordinates {t, x, y, z} defining the interval between arbitrary events constitute a four-vector that undergoes transformations under Lorentz boosts and rotations. Nevertheless, the invariant spacetime separation, s, between these events remains unaltered,

$$s^2 = -t^2 + x^2 + y^2 + z^2 = \; constant \tag{2.19}$$

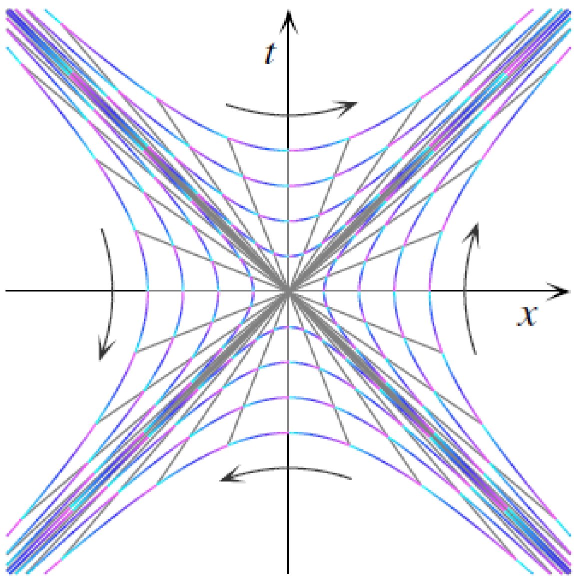

Figure 2.3: Spacetime wheel [3].

2.10 POSSIBLE LIMITATIONS FOR SPECIAL RELATIVITY

Special relativity has fundamentally altered the classical notion of a universal, fixed time, establishing that the passage of time is contingent upon both the observer's reference frame and their spatial position. The theoretical predictions of special relativity have been rigorously confirmed through numerous experimental observations, including phenomena such as the relativity of simultaneity, length contraction, time dilation, the relativistic velocity addition formula, the relativistic Doppler effect, relativistic mass, the universal speed limit, mass-energy equivalence, the finite speed of causality, and Thomas precession, among others. Rather than an invariant time interval, special relativity posits the existence of an invariant space-time interval between two events, which remains constant across all inertial reference frames. The principles of special relativity, combined with other physical postulates and the assertion that mass has inertia (so is subject to at least one additional force) imply a fluctuation between energy and mass in systems which close up, as described by $E = mc^2$, where c is the speed of light. It also synthesizes the relationship between electricity and magnetism [11–13]. The exchange over of Newtonian mechanics and Galilean transformations for Lorentz transformation is one significant characteristic in Special Relativity. Time and space cannot be referenced independently in Galilean relativity, it's unified "space-time" dimension in Special Relativity. Events that happen simultaneously for an observer may take place at different times for another observer. The field of physics that deals with the micro universe is known as quantum physics. Quantum physics and Special Relativity go hand in hand. Relativistic quantum electrodynamics is, in fact, an astonishingly accurate theory. Special Relativity theory, based on current knowledge, is as compatible with experiments in any aspect of physics, and relativity has been able to describe most macro-world occurrences. However, for the micro-world, such as subatomic particles and the Planck length, their principles are ineffective [6]. The Planck length lp is determined by assembling the constants of nature in a manner amenable to yielding an associated length unit. The Planck length and time are extremely small, it is commonly taken as the smallest possible unit of length. As a result, it appears that observers in various frames should view the Planck length and time to be the same, according to the principle of Special Relativity. So, it's question that what is it about length contraction and time dilation. As per Special Relativity's time dilation and length contraction, two observers moving in the same direction are destined to fight about times and lengths for the rest of their lives. This is where the paradox arises: if Planck length is accepted as a minimal length, it contradicts Special Relativity [14, 15]. In particle physics, numerous aspects of particle accelerators are governed by Einstein's theory of Special Relativity, from the kinematics of high-energy particles moving at velocities approaching the speed of light to the time dilation effects that enable the storage of short-lived particles. Particle accelerators are classified by the energy they can generate, and the most recent generation can generate energies of up to 10^{13} eV. But the Planck energy is 1.2×10^{28} eV, which is 10^{15} eV factor higher than the generated energies [14] At the Planck scale, time and space cease to retain their conventional macroscopic meanings, rendering traditional applications of relativity seemingly implausible. In astrophysics, ultra-high-energy cosmic rays are particles, most likely protons, created by distant active galaxies and detected in the atmosphere using particle-physics processes. The Greisen-Zatsepin-Kuzmin (GZK) limit is one element of cosmic-ray measurements that is directly related to relativity. The value of E_{GZK} , the threshold energy, is solely kinematical and it is $E_{GZK} \simeq\ 5 \times 10^{19}$ eV found in Special Relativity [4]. The maximal energy of cosmic ray protons that have traveled extensive distances should be used as the upper limit. It is generated by integrating the principles of energy- momentum conservation with the dispersion relation in a particular relativity theory. However, several cosmic-ray observations contradict this assumption. Cosmic rays with energy greater than 10^{19} eV reach on Earth at a rate of only 1 particle per square kilometer every year, according to The Pierre Auger Observatory**.** In Special Relativity, the observation of these particles was known as the GZK paradox or cosmic-ray conundrum. Here, along with Planck scale and UHECRs there remain important open questions concerning dark matter, unification of Gravity and Quantum Mechanics (QM), potential further dimensions in space-time. Science has come to the same pass 100 years after Einstein invented relativity and quantum mechanics. A rethinking should be necessitated of our fundamental beliefs. Especially in the realms of the very small (elementary particle physics) and the very huge(cosmology). Perhaps the new generation of TeV particle accelerators envisioned for the future will give instruments to answer some of these concerns and lead to the discovery of new phenomena. In the same way that relativity was a dramatic departure from Newtonian concepts but can now be seen as a logical extension of them, a new theory may have the same characteristics as relativity [16].

2.10.1 SPACE, TIME AND GRAVITATION

The special theory of relativity, based on the idea that the laws governing the universe remain consistent for all observers in relative inertial motion, has greatly enhanced our comprehension of essential principles. But applying this to Maxwell's equations, the laws of electromagnetism in his special theory of relativity led him rearrange nearly everything an observer does and how they measure space and time. This invalidated the Newtonian ideas of absolute space and time, as well as all Galileo's transformation: instead came the Lorentz transformation. Although those triumphs, Einstein thought that extraordinary theory simply tackled negligible matters including what's dumped when limbs of reference shred at a fresh clip. He took them to require a more general principle that would prolong their persistence under the laws in question. A fundamental aspect of special relativity is electromagnetic theory, especially the constancy of the speed of light for all inertial observers and the invariance of Maxwell's equations under Lorentz transformations. This principle of invariance created a major conflict with Newtonian gravity, which implied instantaneous action at a distance. Hermann Bondi emphasized the tension between the law of gravitation and special relativity by exploring the implications of the Laplace equation,

$$\nabla^2\varphi = 4\pi\rho G \tag{2.20}$$

where, the mass density is represented by ρ and the gravitational potential by φ. Newtonian theory posits that gravitation possesses its own energy, necessitating its inclusion on the right-hand side of the equation. The energy density associated with gravitation can be expressed in a form similar to the following,

$$\rho_\varphi = -\left[(\nabla\varphi)^2 - \dot{\varphi}^2\right]/(8\pi G) \tag{2.21}$$

Adapting Newtonian gravitation to conform to special relativity proved challenging. For instance, the revised (2.20) includes a term dependent on $(\nabla\varphi)^2$ on the right-hand side, which introduces non-linearity into the problem. Additionally, the self-interaction of φ

alters its value, further complicating (2.20). The idea of this discussion is instead to illustrate that the original Newtonian law gets non-linear with respect to special relativity. Since it cannot be isolated in a finite region free from the influence of gravitational forces, Newton's law must be combined with special relativity to take its validity into account. Everywhere there is matter, including in regions devoid of gravity. This is solved by Einstein's general theory of relativity which gives room for the presence ever present gravity. A planet orbits the Sun under an inverse-square force of attraction towards it, as is allowed by Newtonian gravitation with a radial law correlating combined electricity and magnetism through Maxwell's electromagnetic theory to show that light (reflection)is simply another manifestation of surface-bound electromagnetic oscillations. Einstein's special theory of relativity is unique because nothing else runs at the speed of light, quantum mechanics by its uncertainty principle is different from classical mechanics. So, let us compare general relativity with electricity to see what makes the former unique. We are aware of the Coulomb inverse square law (for electric charges; it is exactly similar to Newton's gravitational one), that unlike two electric charges attract each other but not gravitationally. This much we know, but no further, since parallel to gravitation electric charges of like sign will repel one another. Gravitation is ever-present and forms an integral part of Einstein's general theory of relativity. He claimed that as a consequence of its indefeasibility (not being firmly established), gravitation must be associated with an invariant feature of space and time, and just like the action itself. He recognized this quality as the geometry of space and time, and he proposed that anything with gravitation-like effects happens because the area-time geometry is strange. This understanding of it and how this leads to gravitational effects separates general relativity from other physical theories.

2.10.2 NON-EUCLIDEAN GEOMETRIES

Founded by Greek mathematician Euclid around 300 BC, Euclidean geometry is a logical system in which theorems (attributions that can be formed from previous statements through chains of mathematical reasoning) about triangles, parallelograms and circles are proven based on self-evident postulates. These beliefs are considered to be true by definition and require no "proof". Some of theorems in Euclid's geometry, called parallel postulate are founded on these hypotheses.

In the last century, mathematicians realized that Euclid's postulates were not sacrosanct. Any new set of postulates can yield a novel and internally consistent form of geometry, as demonstrated by mathematicians like Gauss, Bolyai, Lobatchevsky, and Riemann. These new geometries, collectively called non-Euclidean geometries, can be constructed by changing the parallel postulate, for example, one might consider the possibility that no straight line through point P is parallel to line l, or alternatively, that multiple lines through P can be drawn parallel to l. Geometries founded on such altered postulates are categorized as non-Euclidean. In simple words, we can say that Euclidean geometry is a set

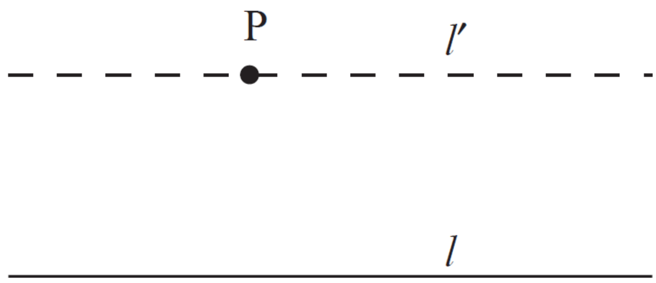

Figure 2.4: The Euclid's parallel postulate [4].

of formal axiomatic and evidentiary theorems which draw upon aspects on theory dealing with triangle construction and parallelograms. On the surface of a sphere, the geometry on the surface of a sphere is non-Euclidean and straight lines are represented by great-circle arcs, and parallel lines do not exist. Every pair of great circles intersects, and no line can be drawn through a point P parallel to a given line l. Consequently, the concept of straightness differs fundamentally from Euclidean geometry, as lines such as latitudes on a flat map of the Earth do not represent the shortest paths but appear curved. This is why aircraft pilots prefer to navigate along great-circle routes, which provide the shortest distance between points. The Newtonian interpretation that gravity is a force was discarded, and Einstein instead defined the effect of gravity on spacetime as giving it a non-Euclidean character which leads to geodesics.

2.10.3 GRAVITY, GEOMETRY AND DYNAMICS

Space and time, or spacetime, are included in the non-Euclidean geometry of space that Einstein proposed [4]. Figure-2.5 illustrates a dynamics scenario in which a stone is hurled vertically upward with a starting velocity of υ. The stone's height at any given time t is determined by,

$$y = \upsilon t - \frac{1}{2} g t^2 \tag{2.22}$$

where, g = acceleration due to gravity. The correct way to interpret this Newtonian equation is: in the absence of gravity, there would be no net external force on the stone and it simply move along a up straight line with speed uniformly. In this case, a dotted straight line corresponds to the trajectory and it is shown like this next image of ideal parabolic continuous line, Einstein saw the continuous trajectory as what needed to be explained. The

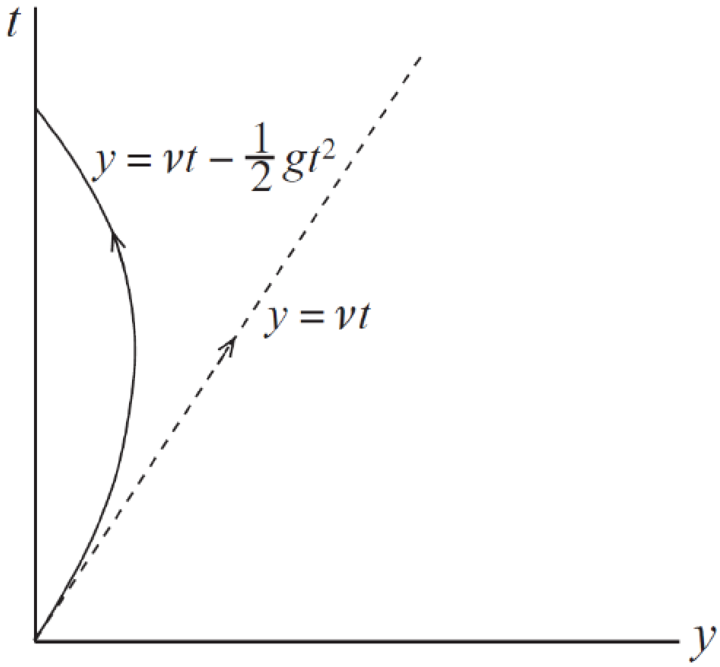

Figure 2.5: The dotted trajectory in Newtonian dynamics denotes straight-line travel without acceleration. On the other hand, the trajectory when acceleration is caused by gravity is depicted by the curved continuous line. Only this curved trajectory, in Einstein's estimation, reflects uniform motion on a straight line within the spacetime that is "curved" by gravity and possesses physical existence [4].

geometry of the world is different, linearity and speed limit are redefined. The curved arcs on the sphere are, for example, real straight lines in the spherical geometry. Persistence in the change of velocity now does not depend on Euclidean measures but can be determined by a non-Euclidean geometry. Einstein gives us the way to think that gravity is not a force.

The trick in these theories is to find a spacetime geometry background relating the matter that, matter under no force moves along straight-line trajectories with constant velocity but different members of this family are non-Euclidean. These ideas require a mathematical apparatus to be described precisely. So, if spacetime is to be curved, it follows that some sort of machinery for telling us how physics works in such a space [17].

Chapter 3: ESSENTIAL CONCEPTS IN GENERAL RELATIVITY

3.1 INTRODUCTION

The basic observation of constant speed of light leads to the theory, called special relativity. The theory mostly centered about the absence of matter distribution, i.e., gravity. A space-time is flat if the Riemann-Christoffel curvature tensor is zero. Like I said earlier, the words "space-time" refers to a set called Minkowski space-time. In particular, general relativity looks at, how space and time are configured when shaped by gravity. In a space-time that is flat, there are Minkowskian coordinate systems for which the space time metric takes its simplest form,

$$ds^2 = \eta_{\mu\vartheta} dx^\mu dx^\vartheta$$

where, $\eta_{\mu\vartheta}$ = diagonal (1, 1, 1, -1), $x^1 = x$, $x^2 = y$, $x^3 = z$, $x^4 = ct$ in the same way as in a Euclidean space there exist Cartesian co-ordinate systems in which the metric has the simplest form,

$$ds^2 = \delta_{\mu\vartheta} dx^\mu dx^\vartheta$$

In the presence of matter space-time structure becomes non-Minkowskian. In this case it becomes a Riemannian space of Minkowskian signature. The configuration of a Riemannian space is established by the metric tensor, which must adhere to the field equations of general relativity.

3.2 PRINCIPLE OF EQUIVALENCE

The equivalence principle links the motion of an object in a gravitational field (i.e., subject to gravity) with the motion of an object not under gravity, but instead observed from a non-inertial frame. A non-inertial frame is akin to some particular gravitational field. Some specific observations led Einstein to the principle of equivalence. Einstein noted that particles of every kind regardless of if they are light quanta or have mass - experience the same degree of acceleration in the gravitational field, according to the equations of falling bodies. Otherwise, the acceleration can be independent of mass in a non-inertial frame. This makes it possible to identify the situations in a non-inertial frame and in a gravitational field, for an explanation of this identification let us compare a man completely enclosed in a lift in two different situations. In the first situation let us suppose that the lift is in empty space far away from any gravitational field and that the lift is being pulled with uniform acceleration from outside by some external agent which the man inside the lift cannot see.

In this situation the man is in a non-inertial frame, because an object which is not acted on by any force will be seen by the man to have an acceleration equal and opposite to that of the lift. Moreover, all objects, independent of mass, will be seen to have the same acceleration. In the second situation let us suppose that the lift is stationary on the surface of a planet, say the Earth, such that the acceleration due to gravity is the same as the acceleration of the lift in the first situation. In this situation also, all masses will have the same acceleration, since gravity causes the same acceleration in all masses according to the laws of falling bodies. Einstein asserted that it is impossible for the man inside the lift to distinguish between the two situations, since the behavior of masses in the two situations is the same. Thus, being in a gravitational field is the same as being in an accelerating non-inertial frame. Therefore, Einstein asserted that a non-inertial frame is equivalent to a gravitational field. This is the gist of the principle of equivalence.

Be aware though that the fields to which non-inertial frames are equivalent do not exactly coincide with real gravity-fields. Outside the sources generating a gravitational field, it is very weak - nearly invisible except at extremely large distances from objects. Nonetheless, this cannot generalize to the case of domains where non-inertial systems make sense. Thus, the fields equal in all absolute accelerations are uniformly valid throughout the space where they refer to systems which undergo accelerated linear motions. If one goes from a non-Minkowskian coordinate system to a Minkowskian then the field in addition vanishes moving into or out of an inertial frame. Coordinates have only been erased, but true gravitational fields cannot be.

3.3 SPACE-TIME METRIC IN A GRAVITATIONAL FIELD

In the previous section we have shown that locally fictitious force and gravitational force are indistinguishable. We have also seen that fictitious force appear when we go to a non-inertial frame. But relative to a non-inertial frame the space-time metric has the general form

$$ds^2 = g_{\mu\vartheta} dx^{\mu} dx^{\vartheta} \tag{3.1}$$

Hence, it follows that the space-time metric assumes the general form (3.1) in the presence of a gravitational field. Now we have to decide if a certain metric, written in the generic form (3.1), comes from a gravitational field or a non-inertial frame. The Riemann-Christoffel curvature tensor sheds light on this difference; it is derived from the metric tensor's second derivatives.

It is important to note that this tensor only equals zero if and only if the metric is a coordinate transformation of the flat metric.

$$ds^2 = \eta_{\mu\vartheta} dx^{\mu} dx^{\vartheta} \tag{3.2}$$

of the Minkowski space-time M_4. Thus, given a metric of the form (3.1) by calculating its Riemann curvature tensor one can tell whether the metric is the co- ordinate transform of a flat metric (3.2) or it is due to the presence of gravitational field.

3.4 PRINCIPLE OF GENERAL COVARIANCE

In the absence of gravitation there exists co-ordinate systems in which the space-time metric has the simplest form-

$$ds^2 = \eta_{\mu\vartheta} dx^{\mu} dx^{\vartheta}$$

takes the general form,

$$ds^2 = g_{\mu\vartheta} dx^{\mu} dx^{\vartheta}$$

Principle of equivalence tells us that, $ds^2 = g_{\mu\vartheta} dx^{\mu} dx^{\vartheta}$ is also the structure of the spacetime metric in the presence of gravitation leads to an intriguing concept. One might begin by considering the form of physical laws expressed in inertial coordinates, then transform these laws into non-inertial coordinates, and finally generalize the resulting expressions to hold for arbitrary coordinates and metrics. This means that we should be able to write any law of physics in a form which is invariant under arbitrary co-ordinate transformations. This is the principle of general covariance. In order to construct generally covariant equations, we need tensors, which transform in a definite way under change of co-ordinate systems.

The laws of physics should have tensorial form because the expression of a law by a tensor equation has exactly the same form in all systems of co-ordinates. For, example the general covariant form of the equation-

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

is the tensor equation,

$$ds^2 = g_{ij}dx^i dx^j \qquad (i, j = 0, 1, 2, 3)$$

where, g_{ij} is a second rank covariant tensor obeying the transformation law,

$$g'_{ij} = \frac{\partial x^{\mu}}{\partial x'^{i}} \frac{\partial x^{\vartheta}}{\partial x'^{j}} g_{\mu\vartheta}$$

Now, consider a certain law of nature in x^i co-ordinate is represented by the tensor equation-

$$A^{\mu}_{\vartheta} = B^{\mu}_{\vartheta} \tag{3.3}$$

In the $\overline{x}^i$ co-ordinate system we have,

$$\overline{A}^{\mu}_{\vartheta} = \frac{\partial \overline{x}^{\mu}}{\partial x^{i}} \frac{\partial x^{j}}{\partial \overline{x}^{\vartheta}} A^{i}_{j}$$

and,

$$\overline{B}^{\mu}_{\vartheta} = \frac{\partial \overline{x}^{\mu}}{\partial x^{i}} \frac{\partial x^{j}}{\partial \overline{x}^{\vartheta}} B^{i}_{j}$$

Subtracting the second from the first we get,

$$\overline{A}^{\mu}_{\vartheta} - \overline{B}^{\mu}_{\vartheta} = \frac{\partial \overline{x}^{\mu}}{\partial x^{i}} \frac{\partial x^{j}}{\partial \overline{x}^{\vartheta}} A^{i}_{j} - \frac{\partial \overline{x}^{\mu}}{\partial x^{i}} \frac{\partial x^{j}}{\partial \overline{x}^{\vartheta}} B^{i}_{j}$$

$$= \frac{\partial \overline{x}^{\mu}}{\partial x^{i}} \frac{\partial x^{j}}{\partial \overline{x}^{\vartheta}} (A^{i}_{j} - B^{i}_{j}) = 0 \qquad \text{as } (A^{i}_{j} - B^{i}_{j}) = 0$$

$$A^{\mu}_{\vartheta} - B^{\mu}_{\vartheta} = 0$$

$$A^{\mu}_{\vartheta} = B^{\mu}_{\vartheta} \tag{3.4}$$

Equation (3.3) and (3.4) are same, only the co-ordinate system is different. Therefore, the law of nature when expressed in the tensor from follow the principle of general covariance.

3.5 FIELD EQUATIONS OF GENERAL RELATIVISTIC BEHAVIOR

In the absence of gravitation, the space-time metric can be expressed as-

$$ds^2 = \eta_{ij} dx^i dx^j \tag{3.5}$$

where, η_{ij} = diagonal (1, -1, -1, -1), $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$ and where the indices i, j run from 0 to 3. In the presence of gravitation, the space-time metric takes the more general form,

$$ds^2 = g_{ij} dx^i dx^j$$

and cannot be reduced to the simpler form (3.5) by any co-ordinate transformation.

It can be shown that knowing the gravitational field allows one to determine the metric tensor g_{ij} of the spacetime manifold, which is fundamentally linked to the field. On the flip side, mass distribution controls the gravitational field. Also, special theory of relativity states that energy and mass are essentially the same thing. Thus, the density of mass and energy must be related to the spacetime manifold's metric tensor g_{ij}. Now, the energy and mass distribution are described by the energy-momentum tensor T_{ij}. Hence, there must exist an equation which relates g_{ij} and T_{ij}. Both the tensors g_{ij} and T_{ij} are of vanishing divergence (i.e.$T^{ij}{}_{;j} = 0$ *and* .$g^{ij}{}_{;j} = 0$) and are symmetric. Hence a possible form of the required equation is,

$$T^{ij} = \lambda g^{ij} \tag{3.6}$$

where, λ is a universal constant. However, in the absence of matter and energy, we have $T^{ij} = 4\times 4$ null matrix whereas g_{ij} must always have some non-zero components. This suggests that the equation (3.6) is incorrect. Now according to Newtonian theory, we have,

$$\nabla^2 V = 4\pi\upsilon\mu \tag{3.7}$$

where V is the gravitational potential, μ is the density of matter and υ is the gravitational constant. The equation that is being sought must include (3.7) as an approximation. But T_{00} involves μ . So, it is expected that the other member of the required equation will provide terms which can approximate $\nabla^2 V$. We know that g_{00} is related to V. This implies there may be second-order derivatives of g_{ij}. As a result, we have to deal with this magical object - the rank-2 symmetric tensor which depend on second derivative g_{ij} and has zero divergence; it should be related to T^{ij}. Extended properties of Einstein (Einstein tensor, etc.) reflects these features. Therefore, we put,

$$R^{ij} - \frac{1}{2}\, g^{ij} R = -k\, T^{ij} \tag{3.8}$$

where k is a constant. Equation (3.8) represents Einstein's gravitational law. By gradually decreasing the indices, it can be stated in two different ways,

$$R^i_j - \frac{1}{2}\, \delta^i_j R = -kT^i_j \tag{3.9}$$

$$R_{ij} - \frac{1}{2} g_{ij} R = -kT_{ij} \tag{3.10}$$

it is obtained from equation (3.9),

$$R^i_i - \frac{1}{2}\, \delta^i_i R = -kT^i_i$$

$$R - \frac{1}{2}.4.R = -kt$$

$$R = kT$$

Then we have.

$$R_{ij} = k(\frac{1}{2} g_{ij} T - T_{ij}) \tag{3.11}$$

Since, $g^{ij}{}_{;j} = 0$, a possible alternative to the law is,

$$R^{ij} - \frac{1}{2}\, g^{ij} R \; - \Lambda g_{ij} = \; -k\; T^{ij} \tag{3.12}$$

where, $'\Lambda'$ is a cosmological constant. Observation shows that even if $'\Lambda'$ is nonzero it must be very small. For empty space $T^{ij} = 0$ so that $T = 0$, then (3.11) give us the result as,

$$R^{ij} = 0$$

which is the field equation in empty space.

3.6 METRIC WITH SPHERICAL SYMMETRY

The most general form of a spherically symmetric metric can be expressed as follows,

$$ds^2 = \; a\, dr^2 + \; r^2(d\theta^2 + \sin^2\theta d\varphi^2) - bc^2\; dt^2 \tag{3.13}$$

where, a and b are function of r only, and η_{ij} = diagonal (1, 1, 1, -1), and $x^1 = a,\; x^2 = r^2,\; x^3 = \; r^2 sin^2\theta,\; x^4 = \; -b$. For the metric (3.13) we have, $g_{ij} = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2\sin^2\theta & 0 \\ 0 & 0 & 0 & -b \end{bmatrix}$ So, g = $\left|g_{ij}\right| \; = \; -abc^2 r^4 \sin^2\theta$ and hence, $g^{11} = \; \frac{1}{a}$,$g^{22} = \frac{1}{r^2}$,$g^{33} = \; \left(\frac{1}{r^2\sin^2\theta}\right)$, $g^{44} = \; -\frac{1}{bc^2}$ and $g^{ij} \; = \; 0$ for i is not =j. Now we will find out the symbols Γ^s_{ij} . we have,

$$\Gamma^s_{ij} \; = \; \frac{1}{2}\, g^{sk}\, \left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

Now,

$$\Gamma^1_{11} = \frac{1}{2}\, g^{1k} \left(\frac{\partial g_{1k}}{\partial x^1} + \frac{\partial g_{1k}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^k} \right)$$

$$= \frac{1}{2}\, g^{11}\, \frac{\partial g_{11}}{\partial x^k}$$

$$= \frac{1}{2}\frac{1}{a}\frac{\partial}{\partial r}\,(a)$$

$$= \frac{a'}{2a}$$

In the same way other components of Γ can be calculated. It can be seen that its non-vanishing components are given by,

$$\Gamma^1_{11} = \frac{a'}{2a}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}$$

$$\Gamma^1_{22} = -\frac{r}{a}$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}$$

$$\Gamma^1_{33} = -\frac{r}{a}\, \sin^2\theta$$

$$\Gamma^4_{14} = \Gamma^4_{41} = \frac{b'}{2b}$$

$$\Gamma^1_{44} = \frac{c^2 b'}{2a}$$

$$\Gamma^3_{23} = \Gamma^3_{32} = cot\theta$$

$$\Gamma^2_{33} = -sin\theta cos\theta$$

where primes denote differentiations with respect to r. The Riemann-Christoffel curvature tensor is,

$$R^i_{jkl} = \Gamma^i_{rk}\Gamma^r_{jl} - \Gamma^i_{jl}\Gamma^r_{jk} + \frac{\partial}{\partial x^k}(\Gamma^i_{jl}) - \frac{\partial}{\partial x^l}\,(\Gamma^i_{jk})$$

Putting i=l in the above equation, we get,

$$R_{jk} = R^i_{jkl}$$

$$= \Gamma^i_{rk}\Gamma^r_{jl} - \Gamma^i_{jl}\Gamma^r_{jk} + \frac{\partial}{\partial x^k}(\Gamma^i_{jl}) - \frac{\partial}{\partial x^l}(\Gamma^i_{jk})$$

$$= \Gamma^i_{rk}\Gamma^r_{jl} - \Gamma^r_{jk}\frac{\partial}{\partial x^r}(\log\sqrt{-g}) - \frac{\partial}{\partial x^i}(\Gamma^i_{jk}) + \frac{\partial^2}{\partial x^k \partial x^j}(\log\sqrt{-g})$$

If we put j=k=1 in the above equation, we get,

$$\begin{aligned}
R_{11} &= \Gamma^i_{r1}\Gamma^r_{1l} - \Gamma^r_{11}\frac{\partial}{\partial x^r}(\log\sqrt{-g}) - \frac{\partial}{\partial x^i}(\Gamma^i_{11}) + \frac{\partial^2}{\partial x^1 \partial x^1}(\log\sqrt{-g}) \\
&= \Gamma^i_{11}\Gamma^1_{1l} + \Gamma^i_{21}\Gamma^2_{1l} + \Gamma^i_{31}\Gamma^3_{1l} + \Gamma^i_{41}\Gamma^4_{1l} - \Gamma^1_{11}\frac{\partial}{\partial x^1}(\log\sqrt{-g}) - \\
&\quad \Gamma^2_{11}\frac{\partial}{\partial x^2}(\log\sqrt{-g}) - \Gamma^3_{11}\frac{\partial}{\partial x^3}(\log\sqrt{-g}) - \Gamma^4_{11}\frac{\partial}{\partial x^4}(\log\sqrt{-g}) + \\
&\quad \frac{\partial^2}{\partial x^1 \partial x^1}(\log\sqrt{-g}) - \frac{\partial}{\partial x^1}(\Gamma^1_{11}) - \frac{\partial}{\partial x^2}(\Gamma^2_{11}) - \frac{\partial}{\partial x^3}(\Gamma^3_{11}) - \frac{\partial}{\partial x^4}(\Gamma^4_{11}) \\
&= \Gamma^1_{11}\Gamma^1_{11} + \Gamma^2_{21}\Gamma^2_{12} + \Gamma^3_{31}\Gamma^3_{13} + \Gamma^4_{41}\Gamma^4_{14} - \Gamma^1_{11}\frac{\partial}{\partial r}(\log\sqrt{abc^2r^4\sin^2\theta}) - \\
&\quad \frac{\partial^2}{\partial r \partial r}(\log\sqrt{abc^2r^4\sin^2\theta}) - \frac{\partial}{\partial x^r}(\Gamma^1_{11})
\end{aligned} \tag{3.14}$$

We have,

$$\Gamma^1_{11}\Gamma^1_{11} = \frac{a'^2}{4a^2}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r^2}$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r^2}$$

$$\Gamma^4_{14} = \Gamma^4_{41} = \frac{b'^2}{4b^2}$$

$$\Gamma^1_{11}\frac{\partial}{\partial r}\left\{(\log abc^2r^4\sin^2\theta)^{\frac{1}{2}}\right\} = \frac{a'}{2a}\,\frac{1}{2}\,\frac{\partial}{\partial r}\left\{(\log\sqrt{abc^2r^4\sin^2\theta})\right\}$$

$$= \frac{a'}{a}\left(\frac{1}{r} + \frac{b'}{4b} + \frac{a'}{4a}\right)$$

$$= \frac{a'}{ar} + \frac{a'b'}{4ab} + \frac{a'^2}{4a^2}$$

$$\frac{\partial^2}{\partial r \partial r}\left\{(\log abc^2r^4\sin^2\theta)^{\frac{1}{2}}\right\} = \frac{\partial}{\partial r}\left\{\frac{1}{2}\left(\frac{4}{r} + \frac{a'}{a} + \frac{b'}{b}\right)\right\}$$

$$= \frac{2}{r^2} + \frac{a''}{2a} + \frac{b''}{2b} - \frac{a'^2}{2a^2} - \frac{b'^2}{2b^2}$$

and,

$$\frac{\partial}{\partial x^r}\left(\Gamma^1_{11}\right) = \frac{\partial}{\partial x^r}\left(\frac{a'}{2r}\right)$$

Putting these values in (3.14), we get,

$$R_{11} = \frac{b''}{2b} - \frac{a'}{ar} + \frac{a'b'}{4ab} + \frac{b'^2}{4b^2}$$

Similarly, we get,

$$\left.\begin{aligned} R_{22} &= \tfrac{1}{a} - \tfrac{ra'}{2a^2} + \tfrac{rb'}{2ab} - 1 \\ R_{33} &= R_{22}\, \sin^2\theta \\ R_{44} &= c^2\left(-\tfrac{b''}{2a} - \tfrac{b'}{ar} + \tfrac{a'b'}{4a^2} + \tfrac{b'^2}{4ab}\right) \end{aligned}\right\} \tag{3.15}$$

Other components of R_{ij} vanish.

$$R^{\alpha}_{\beta} = g^{\alpha\vartheta}R_{\beta\vartheta} \tag{3.16}$$

$$R^1_1 = g^{1\vartheta}R_{1\vartheta}$$

$$= g^{11}R_{11} + g^{12}R_{12} + g^{13}R_{13} + g^{14}R_{14} = g^{11}R_{11}$$

Similarly,

$$R^2_2 = g^{22}R_{22}$$

$$R^3_3 = g^{33}R_{33}$$

$$R^4_4 = g^{44}R_{44}$$

Putting $\alpha = \beta$ in (3.16) we get the Ricci scalar,

$$R = R^{\alpha}_{\alpha}$$

$$= R^1_1 + R^2_2 + R^3_3 + R^4_4$$

$$= g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} + g^{44}R_{44} \tag{3.17}$$

Now,

$$g^{11}R_{11} = \frac{1}{a}\left(\frac{b''}{2b} - \frac{a'}{ar} + \frac{a'b'}{4ab} + \frac{b'^2}{4b^2}\right)$$

$$= \left(\frac{b''}{2ab} - \frac{a'}{a^2 r} + \frac{a'b'}{4a^2 b} + \frac{b'^2}{4ab^2}\right)$$

$$g^{22}R_{22} = \frac{1}{r^2}\left(\frac{1}{a} - \frac{ra'}{2a^2} + \frac{rb'}{2ab} - 1\right)$$

$$= \left(\frac{1}{ar^2} - \frac{a'}{2a^2 r} + \frac{b'}{2abr} - \frac{1}{r^2}\right)$$

$$g^{33}R_{33} = \left(\frac{1}{r^2}\right)\left(\frac{1}{a} - \frac{ra'}{2a^2} + \frac{rb'}{2ab} - 1\right)$$

$$= \left(\frac{1}{ar^2} - \frac{a'}{2a^2 r} + \frac{b'}{2abr} - \frac{1}{r^2}\right)$$

$$g^{44}R_{44} = \left(-\frac{1}{bc^2}\right)\left\{c^2\left(-\frac{b''}{2a} - \frac{b'}{ar} + \frac{a'b'}{4a^2} + \frac{b'^2}{4ab}\right)\right\}$$

$$= \left(\frac{b''}{2ab} - \frac{b'}{abr} + \frac{a'b'}{4a^2 b} + \frac{b'^2}{4ab^2}\right)$$

Putting these values in (3.17), we get,

$$R = \frac{b''}{ab} - \frac{b'^2}{2ab^2} - \frac{a'b'}{2a^2 b} - \frac{2a'}{a^2 r} + \frac{2b'}{abr} - \frac{2}{r^2} + \frac{2}{ar^2}$$

Which is the Ricci scalar.

3.7 SCHWARZSCHILD'S SOLUTION

Let us consider a static spherical body with its center at the origin. We want to find the exterior solution of the Einstein's field equation,

$$R_{ij} = k(\frac{1}{2}g_{ij}T - T_{ij}) \tag{3.18}$$

Outside the body the space is empty. Hence, at all points outside the body we have $T_{ij} = 0$ and T= 0. Hence, from (3.19) we get,

$$R_{ij} = 0 \tag{3.19}$$

for the exterior solution. Since the matter distribution is spherically symmetric the space-time metric is also spherically symmetric. For such a metric the non-zero components of R_{ij} have been shown to be given by equations (3.15). Hence, equation (3.19) shows that, for the exterior solution we must have,

$$\frac{b''}{2b} - \frac{a'}{ar} + \frac{a'b'}{4ab} + \frac{b'^2}{4b^2} = 0 \tag{3.20}$$

$$\frac{1}{a} - \frac{ra'}{2a^2} + \frac{rb'}{2ab} - 1 = 0 \tag{3.21}$$

and,

$$\frac{b''}{2a} - \frac{b'}{ar} + \frac{a'b'}{4a^2} + \frac{b'^2}{4ab} = 0 \tag{3.22}$$

Multiplying equation (3.20) by $\frac{1}{a}$ and (3.22) by $\frac{1}{b}$ and then adding, we have,

$$-\frac{a'}{a^2 r} - \frac{b'}{abr} = 0$$

$$a'b + b'a = 0$$

$$\frac{d}{dr}(ab) = 0$$

ab = constant Gravitational field is zero at infinity. Hence at infinity the space-time metric must have the form,

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) - c^2 dt^2$$

Hence $a \to 1,\ b \to 1\ as\ r \to \infty$ and we have to the value, $b = \frac{1}{a}\ and\ a = \frac{1}{b}$. Putting the value of b in (3.21) we get,

$$\frac{\frac{ra'}{a}}{\frac{2a}{a}} - \frac{ra'}{2a^2} + \frac{1}{a} - 1 = 0$$

$$\frac{ra'}{2a} - \frac{ra'}{2a^2} + \frac{1}{a} - 1 = 0$$

$$ra' = a(1-a)$$

$$r\frac{da}{dr} = a(1-a)$$

$$\frac{da}{a(1-a)} = \frac{1}{r}\,dr$$

$$\left\{\frac{1}{a} + \frac{1}{(1-a)}\right\} da = (\frac{1}{r})dr$$

$$\log a - \log(1-a) = \log r - \log(-2m)$$

$$\frac{a}{(1-a)} = \frac{-r}{2m}$$

a (r-2m) = r a = $a = \frac{r}{(r-2m)}$ i.e. $a = (1 - \frac{2m}{r})^{-1}$

where m is the constant of integration. Then,

$$b = (1 - \frac{2m}{r})$$

putting these values of a and b in (3.13) we get,

$$ds^2 = (1 - \frac{2m}{r})^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) - c^2\,(1 - \frac{2m}{r})\,dt^2 \tag{3.23}$$

This metric was first obtained by Schwarzschild and hence (3.23) is known as the Schwarzschild metric.

Chapter 4: A BRIEF OVERVIEW OF PERFECT FLUIDS

4.1 INTRODUCTION

In the interesting situations with which one often deals in astrophysical general relativity, the perfect fluid regard as a possible realistic source of gravitational field. Usually, a fluid is a separate but continuous substance. A continuum is a vast multitude of particles, the individual behavior of which cannot be followed. This way we describe only the top-level properties of a collection, e.g. average or aggregate values but not individual records from this array: The behavior of physical quantities like particle concentration, energy density, momentum density, pressure, and temperature is determined by the collective characteristics of large groups of particles rather than by the location of any single particle.For instance, the dynamics of a body of water and the gravitational field it produces are determined by the average characteristics of vast collections of water molecules, not by the specific location of any single molecule.

Still, these features might change geographically inside the lake: Pressure is higher at the bottom than at the top, for instance, and temperature might also change. Similar fluctuations in density with location exist in the atmosphere, another fluid. This begs the issue of the appropriate scale across which these features should be averaged: the collection of particles must be large enough to ensure that the influence of individual particles is negligible, but still small enough to ensure relative uniformity, so that average velocity, kinetic energy, and the spacing between particles remain consistent across the group. Such a collection is referred to as an "element." Though somewhat abstract, this term is useful for describing the behavior of particles in an averaged sense, where the group is considered to have a uniform density, velocity, or temperature. If no such collection exists, for an ideal gas, the continuum approximation fails. The continuum approximation assigns values such as density and temperature to each element, treating them as locally uniform. Because the elements are considered to be "small", a mathematical representation of this approximation is made by assigning each point properties such as density, temperature and many more. Continuum Fields are then fields that have a value at every location and time, formed simply by concatenating two or more different kinds of field.

This continuum concept now extends both to rocks and gases. Most of the time, a fluid is anything which flows. But this definition is nonspecific, and fails to neatly split solid vs. fluid behavior. Under pressure most solids will flow, at least to some extent. Why is a substance solid? On second thought, it's clear that rigidity is a function of forces in the direction normal to two bodies' respective points of contact. For this continuum to stay rigid, each pair of neighboring elements must push and pull the other without allowing any sliding along their common boundary. A fluid means a material that has the property of almost no resistance to shear forces, known as antislipping cases in comparison with pressure which are majorly opposing this force.

4.2 THE GENERAL STRESS-ENERGY TENSOR

The components of the stress-energy tensor define themselves most well within a given, arbitrary reference frame,

$$\mathbf{T}\left(\widetilde{dx^{\alpha}},\widetilde{dx^{\beta}}\right)=\mathbf{T}^{\alpha\beta}$$

The definition of $T^{\alpha\beta}$ in the equation above is fully general. To gain a clearer understanding, let us analyze it specifically within the Momentarily Comoving Reference Frame (MCRF), where the fluid element is at rest and the particles exhibit no spatial momentum. In this MCRF, we observe the following:

(1) T^{00} = ρ = Density of energy.

(2) T^{0i} represents the flux of energy. While the MCRF does not exhibit any movement, energy can still be transferred through the process of heat conduction. In the MCRF, the heat conduction term is denoted by T^{0i}.

(3) T^{i0} represents the momentum density. In the MCRF, the particles have no net momentum. However, when heat conduction occurs, the moving energy carries associated momentum. Therefore, as we will demonstrate below, $T^{0i} = T^{i0}$.

(4) T^{ij} denotes the momentum flux. This term is both fascinating and important. The phenomenon is known as stress.

4.3 CONSERVATION OF ENERGY AND MOMENTUM

Because T captures the fluid's momentum and energy, it stood to reason that if we were going to use this in order to make a statement on conservation laws for these principles, maintaining expression linearity was important. In Figure-4.1 the fluid element is cubical, and depicted only in a cross-section with respect to x, y where z-direction has been omitted. All surface transfer of energy, here we can sum the flow rate over faces (4) and (2), respectively, as l^2T^{0x} $(x=0)$ and $-l^2T^{0x}$ $(x=0)$. Since T^{0x} is the energy in x going out of volume across face (2) it will be positive and so second term becomes negative. The energy flowing in the y-direction is expressed as $l^2T^{0y}(y=0)-l^2T^{0y}(y=l)$. The net rate of energy flow must be equal to the rate of change of the total energy content, given by $\frac{\partial(T^{00}l^3)}{\partial t}$.

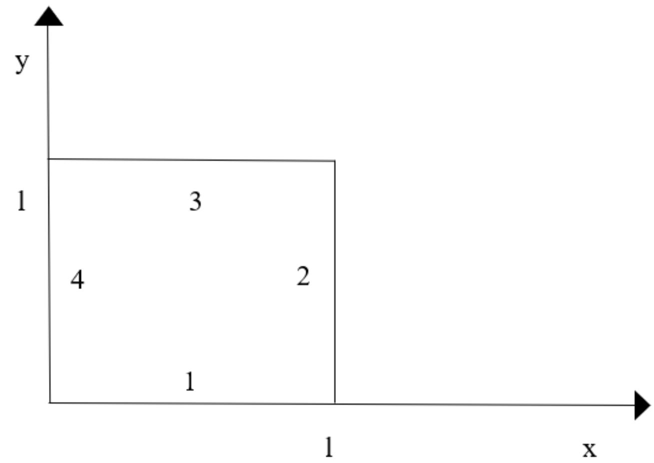

Figure 4.1: A section where z is constant within a cubic fluid element [2].

so, we have,

$$\frac{\partial}{\partial t}\, l^3T^{00} = l^2\left[T^{0x}(x=0)-T^{0x}(x=l)+T^{0y}(y=0)\right.$$
$$\left.-T^{0y}(y=l)+T^{0z}(z=0)-T^{0z}(z=l)\right] \tag{4.1}$$

Diving by l^3 and taking the limit $l\to 0$ gives,

$$\frac{\partial}{\partial t}\,T^{00} = -\frac{\partial}{\partial t}\,T^{0x} - \frac{\partial}{\partial t}\,T^{0y} - \frac{\partial}{\partial t}\,T^{0z} \tag{4.2}$$

We can find from equation (4.2),

$$T^{00}{}_{,0} + T^{0x}{}_{,x} + T^{0y}{}_{,y} + T^{0z}{}_{,z} = 0$$

$$T^{0\alpha}{}_{,\alpha} = 0 \tag{4.3}$$

That is an expression of the conservation of energy principle. Similarly, the momentum conservation principle remains valid. The same mathematical principles apply, with the replacement of index '0' by whatever spatial index goes along that one conserved component of momentum involved. This general conservation law is represented by the equation, then we get,

$$T^{\alpha\beta}{}_{,\beta} = 0 \tag{4.4}$$

This concept is applicable in the context of Special Relativity (SR), and it describes a conservation law related to the Gauss theorem.

4.4 PERFECT FLUIDS

Let us now examine the specific type of fluid that is the focus of our discussion. In the framework of relativity, a perfect fluid is defined by the absence of viscosity and heat conduction within the MCRF. This concept extends the notion of an "ideal gas" from classical thermodynamics. Alongside dust, it represents one of the simplest fluid models to analyze. The stress-energy tensor is significantly simplified under the imposition of these two constraints.

4.4.1. NO THERMAL CONDUCTION

By analyzing **T**, we can deduce that in the Momentarily Comoving Reference Frame , $T^{0i} = T^{i0} = 0$. Energy flow requires particle movement. The first law of thermodynamics let us ascertain that, given a fixed particle count, specific entropy is directly linked to heat transfer according to this law. In a perfect fluid, assuming the conservation of particles is upheld, it follows that the quantity S remains constant over time while the fluid flows. We will observe how this emerges from the conservation laws shortly.

4.4.2. NO VISCOSITY

Viscosity refers to the motion that occurs parallel to the interface between particles. The absence of T^{ij} implies that the forces at the point of contact must always be perpendicular to the interface., which implies that $T^{ij} = 0$ except when i=j, so, T^{ij} is a square matrix with only diagonal elements. Moreover, this diagonalization should take place in all MCRF frames because the idea of 'no viscosity' has nothing to do with any spatial axis. Within each frame, the single diagonal of that matrix is a scalar-multiple of an identity operator which in turn implies all the elements on its main diagonal are equal. So, in summary, x surface will only feel a force either in x-direction or y, z-direction. There are the same number of forces per unit area and we can describe all of them as the pressure, p. What we get out is $T^{ij} = p\delta^{ij}$. By assuming zero viscosity, we have reduced the number of functions from six down to a single function, which is the pressure.

4.4.3. FORM OF STRESS-ENERGY TENSOR (T)

T contains the elements from the MCRF that we just inferred:

$$T^{\alpha\beta} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{4.5}$$

So, we can show that in the MCRF,

$$T^{\alpha\beta} = (\rho + p)U^{\alpha}U^{\beta} + p\eta^{\alpha\beta} \tag{4.6}$$

For instance, if $\alpha = \ \beta = 0$, then, $U^0 = 1, \eta^{00} = -1$ and $T^{\alpha\beta} = \ (\rho + p) - p = \ \rho.$ as in equation. (4.5). We may confirm that equation (4.6) yields equation (4.5) by attempting every conceivable value for α and β. However, equation (4.6) is a frame-invariant formula because it entails,

$$T = (\rho + p)\overrightarrow{U} \bigotimes \overrightarrow{U} + pg^{-1} \tag{4.7}$$

This is the tensor of stress-energy for a fluid displaying ideal properties.

Looking into the Concept of Pressure:

Equation (4.7) in contrast with, $T = \overrightarrow{p} \otimes \overrightarrow{N} = mn\overrightarrow{U} \otimes \overrightarrow{U} = \rho \overrightarrow{U} \otimes \overrightarrow{U}$, In that context, dust is a type of perfect fluid with no pressure. Thus, the pressure would vanish in this case also, since a perfect fluid can only have no pressure if its particles are absolutely at rest. When particles move at unpredictable speeds, pressure is created. Being a gas so rarefied as to be nearly collisionless, it results that even the very tenuous (normal) atmospheric pressure. Pressure is defined as the rate of change in momentum, which is important to understand this concept, irrespective whether it causes due to forces or particles crossing a boundary.

4.5 THE CONSERVATION LAWS FOR PERFECT FLUID

From general conservation law,

$$T^{\alpha\beta}{}_{,\beta} = \ [(\rho + p)U^\alpha U^\beta + p\eta^{\alpha\beta}]_{,\beta} \ = 0 \tag{4.8}$$

There are four equations in above equation. One for each α, now assume that,

$$(nU^\beta)_{,\beta} = 0 \tag{4.9}$$

writing the first term of equation (4.8) as,

$$(\rho + p)U^\alpha U^\beta]_{,\beta} = \left[\frac{(\rho + p)}{n} \, U^\alpha nU^\beta\right]_{,\beta} = \ nU^\beta \left(\frac{(\rho + p)}{n} \, U^\alpha\right)_{,\beta} \tag{4.10}$$

Moreover, $\eta^{\alpha\beta}$ is a constant matrix, so, $\eta^{\alpha\beta}{}_{,\gamma} = 0$. This can also indicate that,

$$U^\alpha{}_{,\beta} U_\alpha = 0 \tag{4.11}$$

The proof of equation (4.11) is,

$$(U^\alpha U_\alpha) = -1, \quad (U^\alpha U_\alpha)_{,\beta} = 0 \tag{4.12}$$

Or,

$$(U^{\alpha}U^{\gamma}\eta_{\alpha\gamma})_{,\beta} = (U^{\alpha}U^{\gamma})_{,\beta}\eta_{\alpha\gamma} = 2U^{\alpha}{}_{,\beta}U^{\gamma}\eta_{\alpha\gamma} \tag{4.13}$$

From the symmetry of we can write the last step as, $\eta_{\alpha\beta}$, that means $U^{\alpha}{}_{,\beta}U^{\gamma}\eta_{\alpha\gamma} = (U^{\alpha}U^{\gamma})_{,\beta}\eta_{\alpha\gamma}$. The last term in equation (4.13) converts to,

$$2U^{\alpha}{}_{,\beta}U_{\alpha,}$$

This equals zero by equation (4.12), confirming that we can apply equation (4.11) as follows. The eq. (4.10) has been modified as,

$$= \eta U^{\beta}\left(\frac{\rho+p}{n}\,U^{\alpha}\right)_{,\beta} + \,p_{,\beta}\eta^{\alpha\beta} = 0 \tag{4.14}$$

Multiplying by U_{α} in the above equation we get,

$$nU^{\beta}U_{\alpha}\left(\frac{\rho+p}{n}\,U^{\alpha}\right)_{,\beta} + \,p_{,\beta}\eta^{\alpha\beta}U_{\alpha} \,= 0 \tag{4.15}$$

The last expression is,

$$p_{,\beta}U^{\beta},$$

The term $\frac{dp}{dr}$ denotes the derivative of p,z with respect to the world line of the fluid element. When the β derivative acts on U^{α}, the first term results in zero. So, we get,

$$U^{\beta}\left[-n\left(\frac{\rho+p}{n}\right)_{,\beta} + p_{,\beta}\right] = 0 \tag{4.16}$$

This equation can be simply written as,

$$\frac{d\rho}{d\tau} - \frac{\rho+p}{n}\,\frac{dn}{d\tau} \,= 0 \tag{4.17}$$

From the thermodynamics law we know that,

$$dp - (\rho+p)\frac{dn}{n} \,= nT\;ds \tag{4.18}$$

or,

$$\Delta q = T\ \Delta s \tag{4.19}$$

Comparing Eq. (4.17) and (4.18),

$$U^{\alpha} S_{,\alpha} \ = \ \frac{dS}{d\tau} \ = 0 \tag{4.20}$$

As a result, the particle-conserving perfect fluid motion respects conservation of particular entropy. This property is known as adiabatic. The flowing entropy has the same value at a fluid element as it does before and after, we do not have to consider now. Nonetheless, it is essential to remember that energy conservations in thermodynamics are the constituent of conservation equations. This Equation (4.8) is parallel along the vector U^{α}. The last three terms in equation (4.8) can be obtained as a similar way. We write equation (4.14),

$$= \ \eta U^{\beta} \left(\frac{\rho + p}{n}\ U^{\alpha} \right)_{,\beta} + \ p_{,\beta} \eta^{\alpha\beta} \ = 0$$

Going to the MCRF, where the value of U^i is zero but $U^i{}_{,\beta}$ is not equal to zero. Within the Momentarily Comoving Reference Frame, The zero component of this equation matches its contraction with U_{α}, as we previously analyzed. Therefore, we only need to focus on the components corresponding to the i-indices,

$$\eta U^{\beta} \left(\frac{\rho + p}{n}\ U^{i} \right)_{,\beta} + \ p_{,\beta} \eta^{i\beta} \ = 0 \tag{4.21}$$

Since $U^i = 0$ the β derivative of $\frac{\rho+p}{n}$ contributes nothing and we get,

$$(\rho + p) U^i{}_{,\beta} U^{\beta} = \ p_{,\beta} \eta^{i\beta} = 0 \tag{4.22}$$

By reducing the index, I simplify this process. Since $\eta_i{}^{\beta} = \delta_i{}^{\beta}$,we found,

$$(\rho + p) U_{i,\beta} U^{\beta} + p_{,i} = 0 \tag{4.23}$$

We bring back that $U_{i,\beta}U^{\beta}$ is the four-acceleration in terms with a_i,

$$(\rho + p)a_i + p_{,i} = 0 \tag{4.24}$$

Anyone acquainted with nonrelativistic fluid dynamics will recognize this as the generalization of,

$$(\rho a + \nabla p = 0) \tag{4.25}$$

where,

$$a = \dot{v} + (v.\nabla)v$$

The only difference being the use of $(\rho + p)$ instead of ρ. In fact, in relativity, the quantity $(\rho + p)$ is essentially a mass density acting as "inertial mass density". larger values of $(\rho + p)$ as dictated by equation (4.24), making it harder to move the object. The equation (4.24) is The basic flow relation $F = ma$ applies here, where $-p_{,i}$ where, p represents the force that a fluid element applies to its adjacent element, while -p signifies the force exerted by the neighboring element back on the fluid element in focus. Nevertheless, the neighbor on the opposite side of the element is pressing back in the opposite direction. It means that only a force occurs, if there is an acceleration which will be produced as a resultant when applying pressures across the fluid element. It gives the force, which is $-\nabla p$.

4.6 SIGNIFICANCE OF GENERAL RELATIVITY

General relativity means a relativistic theory of gravity. Conversely, we were unable to explore it right away due to acceleration without gravity, we had not yet developed a solid understanding of tensors, fluids in special relativity, and curved spacetime. Remember from the introduction that we have not yet discussed curvature but here one can already look forward and see what is going to be developed as a theory of which all those words are theories.

The other observation concerns the prioritization of **T** in general relativity (GR). In Newtonian theory, the gravitational field is sourced by the mass density ρ, which we interpret as approximating ρ_0. Relying solely on rest mass as the source in a theory would be challenging from a relativistic standpoint, given the interchangeability between energy and rest mass. In fact, such a theory would be inconsistent with high-precision experimental results. Therefore, the gravitational field should be sourced by the total energy density, T^{00}, which encompasses all forms of energy. Since using only one component of this tensor to source the field would violate covariance, Einstein extended the concept to include stresses, pressures, and momenta as sources as well. This insight, combined with his understanding of curved spacetime, led him to the formulation of general relativity.

The second point concerns pressure, which plays a markedly different role in general relativity compared to Newtonian theory for two reasons: first, because it acts as a source of the gravitational field and secondly, due courtesy from its $(\rho + p)$ quotient term in equation (4.24). Imagine a compact star with an intense gravitational field that necessitates a hefty pressure gradient. How quantified by the acceleration experienced by an unpressurized fluid element, a_i , with respect to scale. In field given, and hence, the pressure gradient is required that simply be what gives exactly deceleration in the absence of gravity.

$$-a_i = \frac{p_{,i}}{(\rho + p)}$$

This leads to the pressure gradient $p_{,i}$. Therefore, the relative value of gradient$(\rho + p)$ must be greater than ρ. Additionally, as every form of energy (component **T**) source the gravitational field larger pressures contribute to the gravity and hence a greater pressure is needed in order for the star not to totally collapse under its own weight (compared with its Newtonian stars). This does not matter for stars that $(p \ll \rho)$. But not when p is of the same order as ρ. It is inevitable that creating a pressure gradient makes the situation worse; no increase in pressure can support the star against gravity, and it must collapse further. This description, of course shortcuts a lot of math that you need to do along the way but it shows us how even when looking at fluids in SR we start coming across some basic changes GR makes to gravitation.

Chapter 5: THE EINSTEIN FIELD EQUATIONS

5.1 INTRODUCTION

We investigate the Einstein field equations, the basic form of general relativity, in this chapter. Key to the theory, these equations define the fundamental link between energy and matter distributions inside spacetime geometry. In this chapter, we shall discuss the physical causes of the field equations and their evolution before then we will consider their linearized form in the weak-field limit. Deriving the Newtonian limit of general relativity will next prove the theory compatible with classical gravity. The chapter will also briefly address the challenges of simulating matter distributions in gravitational fields, especially those related to interior solution and the field equations.

5.2 FOUR-DIMENTIONAL VELOCITY

The four-velocity of a world line is a vector that is of particular significance.Velocity was defined as a vector tangent to the trajectory of a particle in Galileo's three-dimensional framework. In the four-dimensional spacetime geometry of our system, the four-velocity vector $\overrightarrow{U}$ is defined as a vector that is tangent to the particle's world line. This value is selected to precisely represent the one-unit rest frame time interval of the particle. Examine a particle exhibiting consistent motion. The four-velocity is oriented along the time axis and possesses a magnitude of one unit of proper time as observed from an inertial frame where the particle is instantaneously at rest. As a result, it is the same as $\overrightarrow{e_0}$ in that reference frame. Consequently, the four-velocity of a particle in uniform motion can be characterized as the vector $\overrightarrow{e_0}$ within its inertial rest frame.The ordinary velocity υ, often called three-velocity, is directly related to the spatial components of $\overrightarrow{U}$.

In a non-accelerating frame of reference, a particle that is in motion does not remain stationary. Nonetheless, there is an inertial frame that temporarily aligns with the particle's velocity before progressively dissipating momentum. This frame is referred to as the Momentarily Comoving Reference Frame, a concept of significant relevance. For a particular accelerated particle at a defined event, there exists an infinite number of MCRFs.

While these frames maintain a consistent velocity, their spatial axes are interconnected via rotations. This variability is typically of minimal significance.

In a given event, the basis vector $\vec{e_0}$ of an accelerating particle's MCRF determines its four-velocity, with $\vec{e_0}$ being tangential to the particle's curved trajectory. In Figure-5.1, the particle at event $\boldsymbol{A}$ is associated with the MCRF denoted as $\vec{O}$, and the basis vectors of this reference frame are depicted. At this location, the vector $\vec{e_0}$ is equivalent to $\vec{U}$.

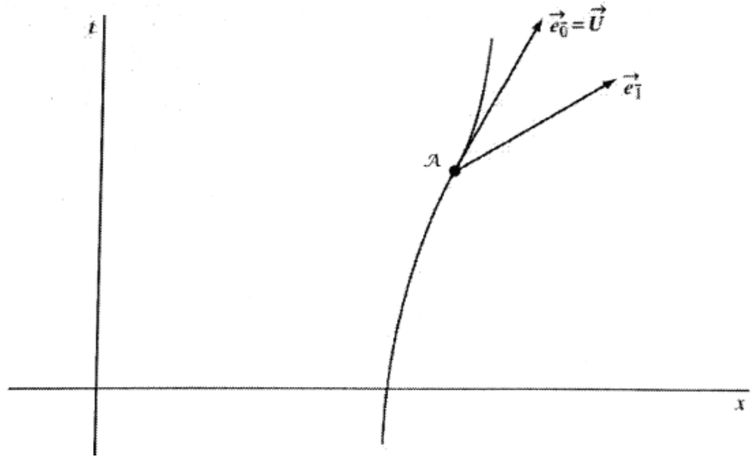

Figure 5.1: At point **A.**, the world line's four-velocity and MCRF basis vectors [2].

5.3 FIELD EQUATIONS: PURPOSE AND JUSTIFICATION

Now that the theory of gravity and its effects on matter has been formalized in terms of a metric-containing curved manifold, the next step is to develop a principle that clarifies how the gravitational field's sources affect the metric. The corresponding equation in Newtonian terms is,

$$\nabla^2 \phi = 4\pi \rho G \tag{5.1}$$

the mass density is represented by ρ. Here is the answer for a point particle whose mass is m:

$$\phi = -\frac{Gm}{r} \tag{5.2}$$

determining the quantity that lacks dimensions when the speed of light, c, equals 1.

In Newtonian mechanics, mass density acts as the primary source of the gravitational field. However, in relativistic gravity, a more comprehensive concept is required, as mass alone does not fully account for relativistic effects. A natural extension in this framework is the total energy, which includes both rest mass and other energy contributions. A fluid element's total energy density is represented by ρ in the Momentarily Comoving Reference Frame. Hence, the relativistic gravitational field might be derived from ρ. But since ρ is the energy density computed from one observer-the MCRF, this method would not be enough. The component T^{00} of the stress-energy tensor helps other observers to depict the energy density. Using ρ as the field's source would suggest a predilection for a given class of observers, namely those for ρ. We do not consider ρ as the source and instead propose that the appropriate generalization of Newtonian mass density is T^{00}. This ensures that all coordinate systems are treated consistently. But there would have to be a specific frame where T^{00} is assessed if it were the only source. The source of the gravitational field can be the full stress-energy tensor $\mathbf{T}$, the theory maintains invariance and avoids favoring any specific coordinate system. Thus, the relativistic generalization of equation (5.1) would be as follows,

$$\mathbf{O(g)} = k\mathbf{T}. \tag{5.3}$$

Here, the differential operator O operates on the metric tensor g, which serves as the generalization of ϕ, with k being a constant. Equation (5.1) will be replaced by ten differential equations, one for each of the independent components in equation (5.3). We need to find a second-order differential operator, denoted as O, that generates a tensor with a rank of $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, analogous to the equation (5.1). Since in equation (5.3), it is set equal to the tensor $\mathbf{T}$, which is represented by the matrix $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ .Alternatively, we can state that the components of $\left\{O^{\alpha\beta}\right\}$ must form a $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ tensor and should be expressed as combinations of $g_{\mu\vartheta,\lambda\sigma}$, $g_{\mu\vartheta,\lambda}$, and $g_{\mu\vartheta}$. It has been demonstrated that the Ricci tensor $R^{\alpha\beta}$ satisfies these criteria. In fact, any tensor of this form can be expressed as follows:

$$O^{\alpha\beta} = R^{\alpha\beta} + \mu g^{\alpha\beta} R + \Lambda g^{\alpha\beta} \tag{5.4}$$

If both μ and Λ are constants, then the given conditions are satisfied. In order to establish the value of μ, we can utilize a property of $T^{\alpha\beta}$ that has not been utilised thus far. This property is the need of the Einstein equivalence principle states that energy and momentum are conserved locally,

$$T^{\alpha\beta}{}_{;\beta} = 0$$

Any metric tensor must satisfy this equation. Thus, equation (5.3) suggests that,

$$O^{\alpha\beta}{}_{;\mu} = 0 \tag{5.5}$$

which, for every metric tensor, must also be true. Given that, $g^{\alpha\beta}{}_{;\beta} = 0$.Using equation (5.4), we now determine,

$$(R^{\alpha\beta} + \mu g^{\alpha\beta} R)_{;\beta} = 0 \tag{5.6}$$

Upon comparing this with the Einstein tensor, it becomes evident that in order for equation (5.6) to hold true for any arbitrary $g_{\alpha\beta}$, we must have $\mu = -1/2$. Therefore, this sequence of argument leads us to the equation,

$$G^{\alpha\beta} \ + \Lambda g^{\alpha\beta} = kT^{\alpha\beta} \tag{5.7}$$

Here, given unknown parameters Λ, k can be expressed as index-free form,

$$\mathbf{G} + \Lambda\mathbf{g} = k\mathbf{T} \tag{5.8}$$

Einstein's field equations are also known as the field equations of General Relativity (GR). The following discussion will demonstrate the process of determining constant k by precisely reproducing Newton's gravitational field equation while maintaining the arbitrary value of Λ. Before advancing further, it is important to provide a brief summary of our current progress. Equation (5.7) has been derived by identifying equations that (i) resemble equation (5.1) but are more general, (ii) are independent of any specific coordinate system, (iii) assurance that for every metric tensor, the energy-momentum can be conserved locally. There are other equations that also satisfy these conditions besides equation (5.7); multiple alternatives exist.

Recent technological advancements have enabled precise testing of Einstein's equations, even within the relatively weak gravitational fields in solar system.

Consequently, newly developed competing hypotheses have come out. These alternates, although crafted to be compatible with Einstein's forecasts to an extent compatible with solar-system investigations, are capable of addressing much stronger fields. Generally, these competing theories are more complex than Einstein's equations and are unlikely to gain significant traction among physicists unless experimental evidence eventually disproves Einstein's framework. The works of Misner et al. and will [18–20] provide detailed accounts of various theories that have been rigorously tested and ruled out through precise experiments since the 1960s.

5.4 WEAK-FIELD EINSTEIN EQUATIONS

Even if Einstein's theory is currently essentially uncontested, there are still good reasons to believe that it will eventually be refuted and to carry out additional experimental research on it. Of course, no quantum theory can be derived from Einstein's theory, and there are currently substantial theoretical attempts underway to develop a coherent quantum theory of gravity. A different kind of gravitational waves could exist in the field itself.

In some such theory, it is possible that any of the general relativity's predictions will be broken. A quantum theory of gravity could one day be guided in its theoretical evolution by the crucial information gleaned from precision experiments on gravitation. Despite their potential interest, these kinds of questions are not covered in this introduction.

Due to the lack of gravity, spacetime becomes flat. A weak gravitational field refers to a situation where spacetime is almost flat. The systems in which the metric components are defined as,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

As we know that, $ds^2 = (\eta_{\alpha\beta} + h_{\alpha\beta})dx^\alpha dx^\beta$; where $h_{\alpha\beta} \approx 0$. Such coordinates are called nearly Lorentz coordinates. In special relativity, the matrix of a Lorentz transformation is thus,

$$\Lambda^{\overline{\alpha}}_{\beta} = \begin{pmatrix} \gamma & -\upsilon\gamma & 0 & 0 \\ -\upsilon\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma = (1-\upsilon^2)^{\frac{-1}{2}}$$

In the context of weak gravitational forces, a background Lorentz transformation can be defined as a transformation that takes the form,

$$x^{\overline{\alpha}} = \Lambda^{\overline{\alpha}}_{\beta} x^{\beta}$$

It is evident that, $h_{\alpha\beta}$ undergoes a transformation analogous to that of a tensor in special relativity when subjected to a background Lorentz transformation. It turns out that we can regard $h_{\alpha\beta}$ as a tensor in special relativity. So, we can define index-raised quantities,

$$h^{\mu}_{\beta} = \eta^{\mu\alpha} h_{\alpha\beta}$$

$$h^{\mu\gamma} = \eta^{\nu\beta} h^{\mu}_{\beta}$$

$$h \equiv h^{\alpha}_{\alpha}$$

let us also write,

$$h^{\alpha\beta} = h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta} h$$

It can be seen that,

$$\overline{h} \equiv \overline{h}^{\alpha}_{\alpha} = -h$$

Moreover, we have,

$$h^{\alpha\beta} = \overline{h}^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}\overline{h}$$

With these definitions Einstein's tensor is given by,

$$G_{\alpha\beta} = \frac{1}{2}\left[\overline{h}^{,\mu}_{\alpha\beta,\mu} + \eta_{\alpha\beta}\overline{h}^{,\mu\nu}_{\mu\nu} - \overline{h}^{,\mu}_{\alpha\mu,\beta} - \overline{h}^{,\mu}_{\beta\mu,\alpha} + 0(h^2_{\alpha\beta}) \right] \tag{5.9}$$

Clearly (5.9) would simplify if,

$$\overline{h}^{\mu\nu}_{,\nu} = 0 \tag{5.10}$$

We call this condition (5.10) the Lorentz gauge condition. In this gauge we have,

$$G^{\alpha\beta} = -\frac{1}{2}\,\Box\,\overline{h}^{\alpha\beta}$$

where,

$$\Box = -\frac{\partial^2}{\partial t^2} + \nabla^2$$

Then the weak field Einstein equations are,

$$\Box\,\overline{h}^{\alpha\beta} = -16\pi T^{\alpha\beta} \tag{5.11}$$

5.5 NEWTONIAN LIMIT

The applicability of Newtonian gravity is restricted to scenarios where gravitational fields are weak enough that velocities do not approach the light speed , i.e., $|\phi| \ll 1$ and $|v| \ll 1$. In this conditions, General Relativity must yield predictions that are consistent with those of Newtonian gravity. Due to the modest magnitudes of velocities, the components $T^{\alpha\beta}$ often satisfy the inequalities$\left|T^{00}\right| \gg \left|T^{0i}\right| \gg \left|T^{ij}\right|$.

'Typically,' is the appropriate term to use because there are certain circumstances, such as a spherical star, where T^{0i} might become zero and the second inequality would not be valid. However, with a Newtonian star with a significant rotation, the value of T^{0i} would surpass all the components of T^{ij} . Now, it is anticipated that these inequalities will apply to $\overline{h}_{\alpha\beta}$ due to the presence of equation (5.11): $\left|\overline{h}^{00}\right| \gg \left|\overline{h}^{0i}\right| \gg \left|\overline{h}^{ij}\right|$. Equation (5.11), in its homogeneous form with the right-hand side set to zero, can have any solution incorporated. Within this solution, the magnitudes of the components of $T^{\alpha\beta}$ are not determined by the magnitudes of the solution's components.

Solutions that exhibit uniform properties throughout are referred to as homogeneous solutions. The ordering presented here for the components of $\overline{h}_{\alpha\beta}$ is valid only when there is no strong gravitational radiation present. Newtonian gravity does not exhibit gravitational waves. Thus, this arrangement is well-suited for accurately reproducing Newtonian gravity within the framework of GR. Consequently, From the main field equation, we may anticipate the prevailing "Newtonian" gravitational field to develop,

$$\Box\, \bar{h}^{00} = -16\pi\rho \tag{5.12}$$

Here, we use the relation $T^{00} = \ \rho + O(\rho\vartheta^2)$ For fields that vary due to the motion of sources at velocity v, we have $\frac{\partial}{\partial t}$ being of the same order as $v\ \frac{\partial}{\partial x}$, so that, we employ the understanding that $T^{00} = \ \rho + O(\rho\vartheta^2)$. When the sources move with velocity v, the change in fields can be described using the equation $\frac{\partial}{\partial t}$ as partial $v\ \frac{\partial}{\partial x}$. So, we can write,

$$\Box = \nabla^2 + O(v^2\nabla^2)$$

Consequently, our equation is, in order of decreasing order, Clearly (5.9) would simplify if,

$$\nabla^2\bar{h}^{00} = -16\pi\rho \tag{5.13}$$

when contrasting this with the Newtonian formula,

$$\nabla^2\phi = -4\pi\rho$$

where,

$$\phi = -\frac{GM}{r}$$

It is evident that we need to identify, (with G =1)

$$\bar{h}^{00} = \ -4\phi \tag{5.14}$$

Given that the other components of $\bar{h}^{\alpha\beta}$ are negligible at this stage, we can therefore conclude that,

$$h = h^{\alpha}_{\alpha} = -\bar{h}^{\alpha}_{\alpha} = \bar{h}^{00}$$

and this implies,

$$h^{00} = \ -2\phi$$

i.e.

$$h_{00} = \eta_{\mu 0}\eta_{\nu 0}h^{\mu\nu} = \eta_{00}\eta_{00}h^{00} = h^{00} = -2\phi$$

$$h^{xx} = h^{yy} = h^{zz} = -2\phi$$

$$ds^2 = \left(\eta_{\alpha\beta} + h_{\alpha\beta}\right) dx^{\alpha} dx^{\beta}$$

$$= (\eta_{00} + h_{00})\, dx^0 dx^0 + (\eta_{11} + h_{11})\, dx^1 dx^1 + (\eta_{22} + h_{22})\, dx^2 dx^2 + (\eta_{33} + h_{33})\, dx^3 dx^3$$

$$= (-1 - 2\phi)dt^2 + (1 - 2\phi)(dx^2 + dy^2 + dz^2)$$

$$= -(1 + 2\phi)dt^2 + (1 - 2\phi)(dx^2 + dy^2 + dz^2) \tag{5.15}$$

Since this measure precisely describes the Newtonian laws of motion, the analysis here shows that this metric can be generated from Einstein's equations, therefore establishing that Newtonian gravity is a particular case of general relativity. This also indicates that the value of k exactly corresponds to the constant 8π in Einstein’s textquotesingle s equations.

For the vast majority of astronomical environments, Newtonian gravity is a good starting point. Nevertheless, post-Newtonian effects corrections that go beyond Newtonian theory must be considered for a great number of systems. Renowned tests that confirm the basic principles of general relativity show that there are post-Newtonian effects in the solar system. The perihelion precession of Mercury and the light's deflection caused by the Sun's gravitational field are two examples. The main post-Newtonian effect outside of our solar system is the Binary Pulsar's orbital contraction, which lends credence to the gravitational radiation predictions of general relativity. The concept of post-Newtonian effects is developed and plays a important role in conducting precise tests of GR. This approximation has been extended to extremely high orders in the literature [20], [21].

5.6 INTERIOR SOLUTION OF EINSTEIN’S FIELD EQUATION

Let us consider a static distribution of matter which exhibits spherical symmetry. The line element having this property in spherical polar co-ordinates can be expressed as,

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{5.16}$$

where υ, λ are functions of r only. i.e. $\lambda = \lambda(r);\ \upsilon = \ \upsilon(r)$ where, $x^0 = t,\ x^1 = r,\ x^2 = \theta,\ x^3 = \varphi$ and the metric (5.16) gives,

$$g_{ab} = \begin{pmatrix} e^{\upsilon} & 0 & 0 & 0 \\ 0 & -e^{\lambda} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}$$

Then,

$$g^{ab} = \begin{pmatrix} e^{-\upsilon} & 0 & 0 & 0 \\ 0 & -e^{-\lambda} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2\sin^2\theta} \end{pmatrix}$$

The Ricci tensor components that are non-zero are as follows:

$$\left.\begin{array}{r} R_{00} = \left[\frac{1}{2}\, e^{\upsilon-\lambda}\left(\upsilon'' + \frac{1}{2}\upsilon'^2 + \frac{1}{2}\upsilon'\lambda' + \frac{2\upsilon'}{r}\right)\right] \\ R_{11} = \left[\frac{1}{2}\upsilon'' - \frac{1}{4}\upsilon\lambda' + \frac{1}{4}\upsilon^2 + \frac{\lambda'}{r}\right] \\ R_{22} = e^{-\lambda}\left[1 + \frac{1}{2}r\upsilon' - \frac{1}{2}r\lambda'\right] - 1 \\ R_{33} = R_{22}\sin^2\theta \end{array}\right\} \tag{5.17}$$

The form of Einstein's field equations when matter is present is,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = -8\pi T_{\mu\nu} \tag{5.18}$$

In the mixed tensor form,

$$R_{\mu}^{\nu} - \frac{1}{2}\delta_{\mu}^{\nu}R + \Lambda\delta_{\mu}^{\nu} = -8\pi T_{\mu}^{\nu} \tag{5.19}$$

Also, we know that,

$$R_{\mu}^{\nu} \ = \ g^{\mu\nu}R_{\mu\nu} \tag{5.20}$$

When $\mu = \nu = 0$, then form (5.23) we get,

$$R_0^0 \ = \ g^{0\alpha}R_{0\alpha}$$

$$= g^{00}R_{00} + g^{01}R_{01} + g^{02}R_{02} + g^{03}R_{03}$$

$$= g^{00}R_{00}$$

Putting the value of g^{00} *and* R_{00} we get,

$$R_0^0 = e^{-\upsilon}\left[\frac{1}{2}e^{\upsilon-\lambda}\left(\upsilon'' + \frac{1}{2}\upsilon'^2 + \frac{1}{2}\upsilon'\lambda' + \frac{2\upsilon'}{r}\right)\right]$$

$$= \frac{1}{2}e^{-\lambda}\left[\upsilon' + \frac{1}{2}\upsilon'^2 + \frac{1}{2}\upsilon'\lambda' + \frac{2\upsilon'}{r}\right] \tag{5.21}$$

Similarly,

$$\left.\begin{aligned} R_1^1 &= -e^{-\lambda}\left[\tfrac{1}{2}\upsilon'' - \tfrac{1}{4}\upsilon\lambda' + \tfrac{1}{4}\upsilon^2 + \tfrac{\lambda'}{r}\right] \\ R_2^2 &= -\tfrac{1}{r^2}e^{-\lambda}\left[1 + \tfrac{1}{2}r\upsilon' - \tfrac{1}{2}r\lambda'\right] + \tfrac{1}{r^2} \\ R_3^3 &= -\tfrac{1}{r^2\sin^2\theta}\left\{e^{-\lambda}\left[1 + \tfrac{1}{2}r\upsilon' - \tfrac{1}{2}r\lambda'\right] - 1\right\}\sin^2\theta \end{aligned}\right\} \tag{5.22}$$

From R_3^3,we can get the value as follow,

$$R_3^3 = -\frac{1}{r^2}e^{-\lambda}\left[1 + \frac{1}{2}r\upsilon' - \frac{1}{2}r\lambda'\right] + \frac{1}{r^2} \tag{5.23}$$

Then the scalar curvature,

$$R = R_\mu^\mu = R_0^0 + R_1^1 + R_2^2 + R_3^3$$

$$R = -e^{-\lambda}\left[\upsilon'' + \frac{1}{2}\upsilon'^2 + \frac{1}{2}\upsilon\lambda' + 2\frac{\upsilon' - \lambda'}{\upsilon} + \frac{2}{r^2}\right] + \frac{2}{r^2} \tag{5.24}$$

Hence the static spherically symmetric line element are,

$$\left.\begin{aligned} R_0^0 - \tfrac{1}{2}\delta_0^0 R &= -e^{-\lambda}\left(\tfrac{\lambda'}{r} - \tfrac{1}{r^2}\right) - \tfrac{1}{r^2} \\ R_1^1 - \tfrac{1}{2}\delta_1^1 R &= e^{-\lambda}\left(\tfrac{\lambda'}{r} - \tfrac{1}{r^2}\right) - \tfrac{1}{r^2} \\ R_2^2 - \tfrac{1}{2}\delta_2^2 R &= e^{-\lambda}\left(\tfrac{1}{2}\upsilon'' + \tfrac{1}{4}\upsilon'^2 - \tfrac{1}{4}\upsilon'\lambda' + \tfrac{1}{2}\,\tfrac{\upsilon' - \lambda'}{r}\right) \\ R_3^3 - \tfrac{1}{2}\delta_3^3 R &= e^{-\lambda}\left(\tfrac{1}{2}\upsilon'' + \tfrac{1}{4}\upsilon'^2 - \tfrac{1}{4}\upsilon'\lambda' + \tfrac{1}{2}\,\tfrac{\upsilon' - \lambda'}{r}\right) \end{aligned}\right\} \tag{5.25}$$

$$R = -e^{-\lambda}\left[\upsilon'' + \frac{1}{2}\upsilon'^2 + \frac{1}{2}\upsilon\lambda' + 2\frac{\upsilon' - \lambda'}{\upsilon} + \frac{2}{r^2}\right] + \frac{2}{r^2} \tag{5.26}$$

From equation (5.22) and (5.25) we get,

$$8\pi T_0^0 = -e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} - \Lambda \tag{5.27}$$

In the same way from (5.22) and (5.25) we get,

$$8\pi T_1^1 = -e^{-\lambda}\left(\frac{\lambda'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} - \Lambda \tag{5.28}$$

Also, from (5.22) and (5.25) we get,

$$8\pi T_2^2 = -e^{-\lambda}\left(\frac{1}{2}\upsilon'' + \frac{1}{4}\upsilon'^2 - \frac{1}{4}\upsilon'\lambda' + \frac{1}{2}\,\frac{\upsilon' - \lambda'}{r}\right) - \Lambda \tag{5.29}$$

And from (5.22) and (5.25) we get,

$$8\pi T_3^3 = -e^{-\lambda}\left(\frac{1}{2}\upsilon'' + \frac{1}{4}\upsilon'^2 - \frac{1}{4}\upsilon'\lambda' + \frac{1}{2}\,\frac{\upsilon' - \lambda'}{r}\right) - \Lambda \tag{5.30}$$

The energy-momentum tensor $T^{\mu\nu}$ for a perfect fluid can be written in terms of the corresponding density ρ and pressure p as:

$$T^{\mu\upsilon} = (\rho + p)\,\frac{dx^{\mu}}{ds}\frac{dx^{\upsilon}}{ds} - g^{\mu\upsilon}p \tag{5.31}$$

On lowering the index, we get,

$$T_{\mu}^{\upsilon} = (\rho + p) g_{\alpha\mu}\,\frac{dx^{\alpha}}{ds}\frac{dx^{\upsilon}}{ds} - g_{\mu}^{\upsilon}p \tag{5.32}$$

Since the distribution of mass is static, all velocity components of fluid matter must be zero i.e.

$$\frac{dr}{ds} = \frac{d\theta}{ds} = \frac{d\varphi}{ds} = 0$$

Then we get from (5.21),

$$\frac{dt}{ds} = e^{\frac{-\upsilon}{2}} \tag{5.33}$$

Now the tensor T_{μ}^{υ} can be written as-

$$T_{\mu}^{\upsilon} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix}$$

i.e. $T_0^0 = \rho,\ T_1^1 = T_2^2 = T_3^3 = -p$, hence equations (5.27), (5.28), (5.29), and (5.30) gets the following forms,

$$8\pi\rho = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} - \Lambda \tag{5.34}$$

$$8\pi p = e^{-\lambda}\left(\frac{\lambda'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} + \Lambda \tag{5.35}$$

$$8\pi p = e^{-\lambda}\left(\frac{1}{2}\upsilon'' + \frac{1}{4}\upsilon'^2 - \frac{1}{4}\upsilon'\lambda' + \frac{1}{2}\,\frac{\upsilon' - \lambda'}{r}\right) + \Lambda \tag{5.36}$$

$$8\pi p = e^{-\lambda}\left(\frac{1}{2}\upsilon'' + \frac{1}{4}\upsilon'^2 - \frac{1}{4}\upsilon'\lambda' + \frac{1}{2}\,\frac{\upsilon' - \lambda'}{r}\right) + \Lambda \tag{5.37}$$

Adding equations (5.35) and (5.37),

$$8\pi(\rho + p) = e^{-\lambda}\left(\frac{\upsilon'}{r} + \frac{\lambda'}{r}\right) \tag{5.38}$$

Differentiating (5.35) with respect to r we get,

$$8\pi\frac{dp}{dr} = e^{-\lambda}\left(\frac{\upsilon''}{r} - \frac{2\upsilon'}{r^2 r^3}\right) - e^{-\lambda}\lambda'\left(\frac{\upsilon'}{r^2} + \frac{1}{r^2}\right) + \frac{2}{r^2} \tag{5.39}$$

Now equating (5.36) and (5.37) we get,

$$e^{-\lambda}\left(\frac{1}{2}\upsilon'' + \frac{1}{4}\upsilon'^2 - \frac{1}{4}\upsilon'\lambda' + \frac{1}{2}\,\frac{\upsilon' - \lambda'}{r}\right) + \Lambda = e^{-\lambda}\left(\frac{\upsilon'}{r^2} + \frac{1}{r^2}\right) - \frac{1}{r^2} + \Lambda \tag{5.40}$$

i.e.

$$e^{-\lambda}\left(\frac{1}{2}\upsilon'' + \frac{1}{4}\upsilon'^2 - \frac{1}{4}\upsilon'\lambda' + \frac{1}{2}\,\frac{\upsilon' - \lambda'}{r} - \frac{\upsilon'}{r^2} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 0 \tag{5.41}$$

multiplying the above equation by $\frac{2}{r}$ and rearranging we get,

$$e^{-\lambda}\left(\frac{1}{r}\upsilon''+\frac{1}{2r}\upsilon'^2-\frac{1}{2r}\upsilon'\lambda'+\frac{\upsilon'-\lambda'}{r^2}-\frac{2\upsilon'}{r^2}-\frac{1}{r^3}\right)+\frac{2}{r^3}=0$$

$$\Longrightarrow e^{-\lambda}\left(\frac{1}{r}\upsilon''+\frac{1}{2r}\upsilon'^2-\frac{1}{2r}\upsilon'\lambda'+\frac{\upsilon'}{r^2}-\frac{\lambda'}{r^2}-\frac{2\upsilon'}{r^2}-\frac{1}{r^3}\right)+\frac{2}{r^3}=0$$

$$\Longrightarrow e^{-\lambda}\left(\frac{1}{r}\upsilon''-\frac{\upsilon'}{r^2}-\frac{2}{r^3}\right)-e^{-\lambda}\lambda'\left(\frac{\upsilon'}{r}+\frac{1}{r^2}\right)+\left(\frac{2}{r^3}\right)+e^{-\lambda}\left(\frac{\upsilon'}{r}+\frac{\lambda'}{r}\right)\frac{\upsilon'}{2}=0 \tag{5.42}$$

With the help of equations (5.38) and (5.39), the equation (5.41) becomes,

$$8\pi\frac{dp}{dr}+8\pi(\rho+p)\,\frac{\upsilon'}{2}=0$$

$$\Longrightarrow \frac{dp}{dr} = -\frac{(\rho+p)}{2}\,\upsilon' \tag{5.43}$$

The solution is intrinsically dependent on the properties of the fluid that the sphere is made of. Schwarzschild resolved this issue by making the assumption that the sphere consists of an incompressible perfect fluid with a proper density of ρ and a proper pressure of p. The equations governing the line element inside the sphere of a compressive fluid are given by equation (5.34), (5.35), (5.36), (5.37) and (5.38). The solution of these equations must satisfy the following boundary conditions,

1) The pressure at the boundary of the sphere is zero.

2) The density ρ is uniform throughout the sphere.

From equation (5.37) we have,

$$e^{-\lambda}\left(\lambda' r-1\right)+1 = (8\pi\rho+\Lambda)r^2$$

$$\Longrightarrow \frac{d}{dr}\left(re^{-\lambda}\right) = 1-(8\pi\rho+\Lambda)r^2$$

Integrating the above equation yields,

$$re^{-\lambda} = r-(8\pi\rho+\Lambda)\frac{r^3}{3}+\alpha$$

(α being integrating constant)

$$\implies e^{-\lambda} = 1 - \frac{1}{3}(8\pi\rho + \Lambda)r^2 + \frac{\alpha}{r}$$

Since at $r \to 0$, $\frac{\alpha}{r} \to \infty$, therefore to remove singularity at r=0 we put $\alpha = 0$. Thus, we have,

$$e^{-\lambda} = 1 - \frac{1}{3}(8\pi\rho + \Lambda)r^2$$

$$e^{-\lambda} = 1 - \frac{r^2}{{R_0}^2} \; where, \; {R_0}^2 = \frac{3}{8\pi\rho + \Lambda} \tag{5.44}$$

Now we have to find the solution for υ. From equation (5.43) we have,

$$\frac{dp}{\rho + p} = -\frac{d\upsilon}{2}$$

As p is constant, integration of above equation yields

$$\log(\rho + p) = -\frac{\upsilon}{2} + \log\beta$$

where, $\log\beta$ is constant of integration, i.e. $(\rho + p) = \beta e^{\frac{-\upsilon}{2}}$

$$8\pi(\rho + p) = 8\pi\beta e^{\frac{-\upsilon}{2}}$$

Using equation (5.38) and putting $8\pi\beta = c$ we get,

$$e^{\frac{-\upsilon}{2}} e^{-\lambda}\left(\frac{\upsilon'}{r} + \frac{\lambda'}{r}\right) = c \tag{5.45}$$

Equation (5.23) can be written as,

$$e^{\frac{\upsilon}{2}}\left(e^{-\lambda}.\frac{\upsilon'}{r} + \frac{2}{{R_0}^2}\right) = c$$

Now substituting the value of $e^{-\lambda}$ from equation (5.44) we get,

$$e^{\frac{\upsilon}{2}}\left[\frac{2}{{R_0}^2} + \frac{\upsilon'}{r}\left(1 - \frac{r^2}{{R_0}^2}\right)\right] = c \tag{5.46}$$

To solve the above equation, we substitute

$$\Psi(r) \; = \; e^{\frac{\upsilon}{2}} \tag{5.47}$$

So that,

$$\Psi'(r) = \; \frac{1}{2} \; \upsilon' e^{\frac{\upsilon}{2}}$$

Hence the equation (5.24) yields,

$$\left(1 - \; \frac{r^2}{{R_0}^2}\right)\Psi'(r) + \frac{r^2}{{R_0}^2}\Psi(r) \; = \frac{1}{2} rc$$

$$\Rightarrow \quad \Psi'(r) + \; \frac{r}{{R_0}^2 - r^2} \; \Psi(r) \; = \; \frac{rc_1}{{R_0}^2 - r^2} \tag{5.48}$$

where constant $c_1 = \; \frac{1}{2} c{R_0}^2$; since the solution of,

$$\Psi' + \Psi p(r) = Q(r)$$

i.e. $(exp.(\int pdr) \; = \; \int Q \; exp.(\int pdr) \; dr \; + \; constant$, Therefore, the solution of equation (5.48) is, $\Psi \; exp.\left(\int \frac{r}{{R_0}^2 - r^2} \; dr\right) \; = \; \int \frac{rc_1}{{R_0}^2 - r^2} \; exp. \; \left(\int \frac{r}{{R_0}^2 - r^2} \; dr\right) dr + c_2$

$$\Rightarrow \quad \frac{\Psi}{\sqrt{{R_0}^2 - r^2}} \; = \; \frac{c_1}{\sqrt{R^2 - r^2}} \; + c_2$$

$$\Rightarrow \quad \Psi = \; c_1 + c_2 \sqrt{{R_0}^2 - r^2}$$

i.e.

$$\Psi(r) \; = \; e^{\frac{\upsilon}{2}} \; = \; A - B\sqrt{1 - \; \frac{r^2}{{R_0}^2}} \tag{5.49}$$

where $A = c_1$, $B = \; -c_2 R_0$ finally we have, $e^{-\lambda} \; = \; 1 - \; \frac{r^2}{{R_0}^2} \;\; e^{\frac{\upsilon}{2}} \; = \; A - B\sqrt{1 - \; \frac{r^2}{{R_0}^2}}$ (where, $\frac{1}{{R_0}^2} \; = \; \frac{8\pi\rho + \Lambda}{3}$) Hence the interior solution of Einstein's equation is given by,

$$ds^2 = -\left(1 - \; \frac{r^2}{{R_0}^2}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) + \left[A - B\sqrt{1 - \; \frac{r^2}{{R_0}^2}}\right]^2 dt^2$$

Chapter 6: SPHERICALLY SYMMETRIC PERFECT FLUID SPHERE

6.1 INTRODUCTION

The spherically symmetric perfect fluid sphere is the most fundamental models in general relativity. Following a quick review of spherical coordinates in flat space, the subject moves to the effects of curved spacetime on two-dimensional surfaces. This chapter explores the features of spherically symmetric spacetimes with a view towards stationary configurations in particular. We investigate the significance of metric terms inside this framework in order to better grasp the effect of symmetry and curvature on the relativistic behavior of perfect fluid spheres.

6.2 FLAT SPACE WITHIN SPHERICAL COORDINATES

One way to express the line element of Minkowski space is by introducing the standard coordinates (r, θ, φ).

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{6.1}$$

Each surface with fixed values of r and t forms a sphere, more precisely a two-sphere, which is a two-dimensional spherical surface. The equation above describes the distances along paths restricted to such a spherical surface, where $dt = dr = 0$.

$$dl^2 = r^2(d\theta^2 + \sin^2\theta d\varphi^2) := \ r^2 d\Omega^2 \tag{6.2}$$

The notation $d\Omega^2$ denotes an element of solid angle. A sphere is characterized by a circumference of $2\pi r$ and an area of $4\pi r^2$. Conversely, any surface with a line element given by equation (6.2), where r^2 is independent of the angular coordinates θ and φ, inherently possesses the geometry of a two-sphere [2].

6.3 CURVED SPACETIME AND ITS IMPLICATIONS ON TWO-SPACE

The spherical symmetry of a spacetime implies that each point within it is situated on a two-dimensional surface that resembles a two-sphere. In other words, the line element of this two-sphere is as follows,

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{6.3}$$

Consider a function $f(r', t)$ of the extra coordinates r' and t on our manifold, which determines the surface area of each sphere as $4\pi f(r', t)$. In this spherical geometry, we define the radial coordinate r by setting $f(r', t) = r^2$, effectively transforming the coordinates from (r', t) to (r, t). Consequently, any surface of constant r and t forms a two-sphere, characterized by a circumference of $2\pi r$ and an area of $4\pi r^2$. This coordinate is known as the 'curvature' or 'area coordinate' since it specifies both the radius of curvature and the area of the spheres.

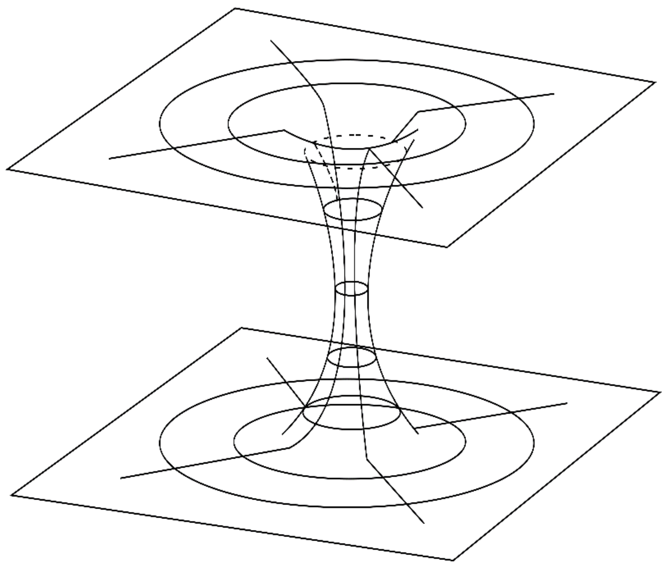

Figure 6.1: The two-space does not contain the center of any circle, but it does contain two plane sheets joined by a circular throat, and there is circular (axial) symmetry [2].

However, there is no inherent or predefined relationship in relation to the sphere's surface and the distance between its center and the variable r. The properties of the spheres themselves are the only factors that form this r. Given that the 'centers' of the spheres in a spacetime are not actual locations on the spheres themselves, the condition of spherically symmetric does not necessarily imply the existence of a point at the centre (at r = 0 in flat space). A clear counterexample of a two-dimensional space containing circles without a central point is illustrated in Figure-6.1. With respect to an axis that passes through the midway of the throat, the entire structure is symmetrical, and it is composed of two sheets joined by a throat.However, the two-dimensional surface shown does not include the axial points that would serve as the centers of the circles [22, 23]. When φ is defined as the angular coordinate around the axis, the line element along each circle is given by $r^2 d\varphi^2$, where r is a constant that distinguishes each circle. This r is analogous to the radial coordinate employed in our description of spherically symmetric spacetimes.

6.4 CIRCULARLY SYMMETRICAL SPACETIME

Since both the spatial slices at constant t and the entire spacetime are spherically symmetric, any line defined by fixed values of r, θ, and φ must be perpendicular to the two-spheres. If this were not the case, it would imply the presence of a preferred direction in space. This condition ensures that the time basis vector, $\overrightarrow{e_t}$, is orthogonal to the angular basis vectors, $\overrightarrow{e_\theta}$ and $\overrightarrow{e_\varphi}$, which is equivalent to stating that $g_{r\theta} = g_{r\varphi} = 0$. Hence, we have:

$$ds^2 = g_{00}dt^2 + 2g_{0r}drdt + g_{rr}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{6.4}$$

This is the general metric for a spherically symmetric spacetime, with g_{00}, g_{0r}, and g_{rr} expressed as functions of r and t. We have taken advantage of our coordinate freedom to reduce it to its simplest form.

6.5 SPACETIMES THAT ARE STATIC AND SPHERICALLY SYMMETRIC

The metric A stationary system, like a star or black hole, serves as the most straightforward physical model available. When a temporal coordinate is linked to two fundamental attributes, we describe the spacetime as unchanging: (i) The metric components remain invariant with respect to time, and (ii) the geometric structure is unaffected by the reversal of the temporal direction, $t \to -t$. If the second condition holds, then reversing the scene would render it indistinguishable from the original. This does not logically follow from (i), as demonstrated by the example of a rotating star: time reversal changes the direction of rotation, but the metric components remain independent of time. A spacetime that meets condition (i) without the requirement of fulfilling (ii) is termed stationary.

Condition (ii) entails the following consequence. The process of converting coordinates from one system to another, $(t,\ r,\ \theta,\ \varphi) \to (-t,\ r,\ \theta,\ \varphi)$ *has* $\Lambda^0_0 = -1,\ \Lambda^i_j = \delta^i_j$ and we find,

$$g_{\overline{oo}} = (\Lambda^0_{\overline{o}})^2 g_{00} = g_{00}$$

$$g_{\overline{or}} = (\Lambda^0_{\overline{o}})(\Lambda^r_{\overline{r}}) g_{0r} = -g_{0r}$$

$$g_{\overline{rr}} = (\Lambda^r_{\overline{r}})^2 g_{rr} = g_{rr} \tag{6.5}$$

Given that the geometry must remain unchanged, it is required that $g_{0r} \equiv 0$. Therefore, the metric of a static, spherically symmetric spacetime can be written as follows:

$$ds^2 = -e^{2\varphi}dt^2 + e^{2\Lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{6.6}$$

the two unknowns $g_{00}(r)$ and $g_{rr}(r)$ have been replaced by φ(r) and Λ(r). This substitution can be used as long as $g_{00} < 0$ and $g_{rr} > 0$ globally. Below, we will observe that these requirements are valid within stars, but they become invalid for black holes. We have adequate evidence to assume that far distant from stars spacetime is mostly flat. This enables us to apply particular boundary conditions (also known as asymptotic regularity conditions) to Einstein's equations,

$$\lim_{r\to\infty} \varphi(r) = \lim_{r\to\infty} \Lambda(r) = 0 \tag{6.7}$$

we state that spacetime is asymptotically flat under this condition.

6.6 PHYSICAL INTERPRETATION OF METRIC TERMS

Our coordinate system is deliberately constructed to reflect the intrinsic physical symmetries of spacetime, thereby assigning significant physical meaning to the metric components. The precise angular separation, in radians, between any two radii r_1 and r_2 is given by:

$$l_{12} = \int_{r_1}^{r_2} e^{\Lambda} dr \tag{6.8}$$

on this curve, $dt = d\theta = d\varphi = 0$, because it is a curve.The importance of g_{00} is more important. Any particle moving along a geodesic has a constant momentum component p_0, which may be defined as the constant -E, because the metric is independent of t,

$$p_0 := -E \tag{6.9}$$

However, a stationary local observer within the spacetime, positioned at any given radius r, will measure a distinct energy value. The four-velocity must have $U^i = \frac{dx^i}{d\tau} = 0$, and the condition $\overrightarrow{U}.\overrightarrow{U} = 1$ implies $U^0 = e^{-\varphi}$. So, the energy is measures by,

$$E^* = -\overrightarrow{U}.\overrightarrow{p} = e^{-\varphi}E \tag{6.10}$$

We have established that a particle characterized by a geodesic defined by the constant E has an energy of $e^{-\varphi}E$ when measured by a locally inertial observer at rest within the spacetime framework. Given that $e^{-\varphi} = 1$ in the limit of large distances, it follows that E stands for the energy that an observer far away would measure if the particle were far away; this is also known as the energy at infinity. Every other location where $e^{-\varphi} > 1$ occurs, the particle must be more energetic than any inertial observers it encounters along its course. The kinetic energy a particle gains as it descends through a gravitational field is proportional to this extra energy [24, 25]. Particularly for photons, this is crucial. Think about a photon that is sent out at a distance r_1 and then picked up at another place. The local energy of an object, denoted as $h\upsilon_{em}$, is equal to its frequency in the local inertial frame υ_{em}, as determined by the emission process (for example, by a spectral line), and its conserved constant E is given by $h\upsilon_{em} exp\ [\varphi(r_1)]$. The frequency of a photon is therefore $\frac{E}{h} \equiv \upsilon_{rec} = \upsilon_{em} exp\ [\varphi(r_1)]$ when it reaches the faraway observer and is measured to have energy E. When applied to photons, this connection defines their redshift. So, weefined by,

$$z = \frac{\lambda_{rec} - \lambda_{em}}{\lambda_{em}} = \frac{\upsilon_{em}}{\upsilon_{rec}} - 1 \tag{6.11}$$

is therefore,

$$z = e^{-\varphi(r_1)} - 1 \tag{6.12}$$

According to this vital equation, e^{φ} has physical importance.

The Einstein Tensor: For the metric specified by equation (6.6), it can be demonstrated that the components of the Einstein tensor are given by,

$$\left.\begin{aligned} G_{00} &= \tfrac{1}{r^2} e^{2\varphi} \tfrac{d}{dr}\left[r(1-e^{-2\Lambda})\right] \\ G_{rr} &= -\tfrac{1}{r^2} e^{2\Lambda}(1-e^{-2\Lambda}) + \tfrac{2}{r}\,\varphi' \\ G_{\theta\theta} &= r^2 e^{-2\Lambda}\left[\varphi'' + (\varphi')^2 + \tfrac{\varphi'}{r} - \varphi'\Lambda' - \tfrac{\Lambda'}{r}\right] \\ G_{\varphi\varphi} &= \sin^2\theta\, G_{\theta\theta} \end{aligned}\right\} \tag{6.13}$$

where, $\varphi' = \frac{d\varphi}{dr}$,etc. all other components vanish.

Stress-energy Tensor: Additionally, we compute the fluid's energy-momentum tensor in the static case. The requirement for normalization is,

$$\vec{U}.\vec{U} = -1 \tag{6.14}$$

So, we have,

$$U^0 = e^{-\varphi},\; U_0 = -e^{\varphi} \tag{6.15}$$

Then **T** has components give us,

$$\left.\begin{aligned} T_{00} &= \rho e^{2\varphi} \\ T_{rr} &= p e^{2\Lambda} \\ T_{\theta\theta} &= r^2 p \\ T_{\varphi\varphi} &= \sin^2\theta\, T_{\theta\theta} \end{aligned}\right\} \tag{6.16}$$

all other components vanish.

Einstein Equations: (6.13) and (6.16) provide the (0,0) component of Einstein's equations. It is helpful to replace $\Lambda(r)$ at this point with an alternate unknown function m(r), which is defined as follows:

$$m(r) := \frac{1}{2} r(1-e^{-2\Lambda}) \tag{6.17}$$

or,

$$g_{rr} = e^{2\Lambda} = \left(1-\frac{2m(r)}{r}\right)^{-1} \tag{6.18}$$

Therefore, the equation at the point (0,0) can be deduced,

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho \tag{6.19}$$

This expression resembles Newtonian equation, where m(r) represents the mass with radius r of a sphere. Consequently, in the context of relativity, m(r) is referred to as the mass function. The total energy is not localizable in General Relativity, hence it cannot be construed as the localized mass energy inside radius r. The (r,r) component of the equation, derived from equations (6.13) and (6.16), can be reformulated as follows:

$$\frac{d\varphi}{dr} = \frac{m(r)+4\pi r^3 \rho}{r[r-2m(r)]} \tag{6.20}$$

In the event that we possess for the system an equation of state as, $p = p(\rho)$, then, this eq. with $(\rho + p)\frac{d\varphi}{dr} = -\frac{dp}{dr}$ and equations (6.17) to (6.20) provide The four unknown variables $(\varphi,\ m,\ \rho,\ p)$ are represented by four equations. Additional information is not provided by the Einstein equations' (θ,θ) and (φ,φ) components, because (1) it is evident from equations (6.13), and (6.16) that these two equations are essentially redundant, and (2) the Bianchi identities confirm that these equations are consequences of the conservation laws expressed in equations (6.19) and (6.20).

Chapter 7: GENERATION OF STATIC SPHERICALLY SYMMETRIC SOLUTIONS FOR EINSTEIN'S EQUATION

7.1 INTRODUCTION

Deriving static spherically symmetric solutions to Einstein's field equations is a fundamental aspect of theoretical physics, essential for understanding the interaction of matter and energy in gravitational fields. The first such solution, corresponding to a spherically symmetric mass with constant density, was discovered by Schwarzschild in 1918 [26]. Since then, numerous exact solutions describing static spherically symmetric ideal fluid configurations have been explored [27, 28]. Several solution-generation techniques have been developed to classify and extend the set of perfect fluid solutions in general relativity [29, 30]. These methods frequently rely on the Buchdahl limit [31], which imposes constraints on the equilibrium structure of self-gravitating fluids.

Extensive research has been devoted to obtaining exact solutions for static and spherically symmetric systems. Golovnev provides a comprehensive overview of such solutions within alternative theories of gravity, emphasizing their simplicity and applicability [32]. Similarly, Kiselev introduced new exact solutions to Einstein's equations that describe various forms of matter surrounding black holes, highlighting their relevance in astrophysical contexts [33]. The importance of static spherically symmetric solutions in general relativity has also been underscored by Ali et al., who explored their physical implications [34]. Additionally, Birkhoff's theorem establishes that these solutions remain both static and asymptotically flat under vacuum conditions, reinforcing their fundamental role in gravitational physics.

Beyond their significance in astrophysics, these solutions provide critical insights into cosmology and matter field behavior. A theorem proposed by Salgado outlines a method for constructing exact solutions to Einstein's equations using specific matter field configurations [35]. Such solutions are indispensable for investigating the properties of matter in strong gravitational fields, particularly in the vicinity of black holes. The study of static spherically symmetric solutions continues to be a crucial area of research, offering deeper insights into the fundamental nature of spacetime, the dynamics of black holes, and the broader implications for cosmological theories [36–39].

7.2 FIELD EQUATIONS

The metric of a static, spherically symmetric spacetime in curvature coordinates is expressed as:

$$ds^2 = -e^{2\varphi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{7.1}$$

The nonzero components of Einstein's tensor for the metric given by equation (7.1) are as follows,

$$\left.\begin{aligned} G_{00} &= \tfrac{1}{r^2}e^{2\varphi}\tfrac{d}{dr}\left[r(1-e^{-2\Lambda})\right] \\ G_{rr} &= -\tfrac{1}{r^2}e^{2\Lambda}(1-e^{-2\Lambda}) + \tfrac{2}{r}\,\varphi' \\ G_{\theta\theta} &= r^2e^{-2\Lambda}\left[\varphi'' + (\varphi')^2 + \tfrac{\varphi'}{r} - \varphi'\Lambda' - \tfrac{\Lambda'}{r}\right] \\ G_{\varphi\varphi} &= \sin^2\theta G_{\theta\theta} \end{aligned}\right\} \tag{7.2}$$

In this context, primes denote derivatives with respect to r. We are analyzing the matter content of the fluid sphere, which is assumed to be a perfect fluid. The nonzero components of the energy-momentum tensor $T_{\mu\nu}$ in this scenario are as follows:

$$\left.\begin{aligned} T_{00} &= \rho e^{2\varphi} \\ T_{rr} &= pe^{2\Lambda} \\ T_{\theta\theta} &= pr^2 \\ T_{\varphi\varphi} &= T_{\theta\theta}\sin^2\theta \end{aligned}\right\} \tag{7.3}$$

where, $\rho(r)$ and $p(r)$ represent the density and pressure, respectively, within the fluid sphere. By substituting the values (7.2)-(7.3) into Einstein's equation $G_{\mu\nu} = 8\pi T_{\mu\nu}$, we derive a set of distinct equations,

$$8\pi\rho = \frac{1}{r^2}\frac{d}{dr}r\left(1-e^{-2\Lambda}\right) \tag{7.4}$$

$$8\pi p = -\frac{1}{r^2}\left(1-e^{-2\Lambda}\right) + \frac{2}{r}\,e^{-2\Lambda}\varphi' \tag{7.5}$$

$$8\pi p = r^2 e^{-2\Lambda}\left[\varphi'' + (\varphi')^2 + \frac{\varphi'}{r} - \varphi'\Lambda' - \frac{\Lambda'}{r}\right] \tag{7.6}$$

From (7.5) and (7.6) we obtain the equation,

$$-\frac{1}{r^2}\left(1 - e^{-2\Lambda}\right) + \frac{2}{r}\, e^{-2\Lambda}\varphi' = r^2 e^{-2\Lambda}\ \varphi'' + (\varphi')^2 + \frac{\varphi'}{r} - (\varphi' + \frac{1}{r})\Lambda'$$

$$-\frac{1}{r^2} + \frac{e^{-2\Lambda}}{r^2} + \frac{2}{r}\, e^{-2\Lambda}\varphi' = e^{-2\Lambda}\ r^2\varphi'' + r^2(\varphi')^2 + \varphi' r - (\varphi' + \frac{1}{r})\Lambda' r^2$$

$$-1 + e^{-2\Lambda} + 2re^{-2\Lambda}\varphi' = r^4 e^{-2\Lambda}\varphi'' + (\varphi')^2 + \frac{\varphi'}{r} - \left(\varphi' + \frac{1}{r}\right)\Lambda'$$

$e^{-2\Lambda}$ appears in multiple time,

$$e^{-2\Lambda}\left[1 + 2r\varphi' - r^4\varphi'' + (\varphi')^2 + \frac{\varphi'}{r} - \left(\varphi' + \frac{1}{r}\right)\Lambda'\right] - 1 = 0$$

$$e^{-2\Lambda}\left[1 + 2r\varphi' - r^4\varphi'' + (\varphi')^2 + \frac{\varphi'}{r} - \left(\varphi' + \frac{1}{r}\right)\Lambda'\right] = 1$$

Calculating Λ' and using combine equation we get,

$$r^4 e^{-2\Lambda}\left(\varphi' + \frac{1}{r}\right)\Lambda' = r^4 e^{-2\Lambda}\left(\varphi'' + (\varphi')^2 + \frac{\varphi'}{r}\right) + 1 - e^{-2\Lambda} - 2re^{-2\Lambda}\varphi'$$

$$\Lambda' = \frac{r^4 e^{-2\Lambda}\left(\varphi'' + (\varphi')^2 + \frac{\varphi'}{r}\right) + 1 - e^{-2\Lambda} - 2re^{-2\Lambda}\varphi'}{r^4 e^{-2\Lambda}\left(\varphi' + \frac{1}{r}\right)}$$

Now, $\frac{d}{dr}\, e^{-2\Lambda} = -2\, e^{-2\Lambda}\, \Lambda'$ and we using the value of Λ', putting it in this equation, we get,

$$\frac{d}{dr}\, e^{-2\Lambda} = -2\, e^{-2\Lambda}\, .\frac{r^4 e^{-2\Lambda}\left(\varphi'' + (\varphi')^2 + \frac{\varphi'}{r}\right) + 1 - e^{-2\Lambda} - 2r\, e^{-2\Lambda}\ \varphi'}{r^4 e^{-2\Lambda}\ \left(\varphi' + \frac{1}{r}\right)}$$

$$\frac{d}{dr}\, e^{-2\Lambda} = -2\, e^{-2\Lambda}\frac{\left(r^2\varphi'' + r^2(\varphi')^2 + r\varphi' - \frac{1}{r^2}\right)}{r^2\left(\varphi' + \frac{1}{r}\right)} - \frac{2\, e^{-4\Lambda}}{r^2\left(\varphi' + \frac{1}{r}\right)} - \frac{4\, e^{-4\Lambda} r\varphi'}{r^2\left(\varphi' + \frac{1}{r}\right)}$$

$$\frac{d}{dr}\, e^{-2\Lambda} = -2\, e^{-2\Lambda}\, .\frac{\left(r^2\varphi'' + r^2(\varphi')^2 + r\varphi' - \frac{1}{r^2}\right)}{r^2\left(\varphi' + \frac{1}{r}\right)} - \frac{2\, e^{-2\Lambda} + 4\, e^{-2\Lambda} r\varphi'}{r^2\left(\varphi' + \frac{1}{r}\right)}$$

$$\frac{d}{dr}\,e^{-2\Lambda} = -2\,e^{-2\Lambda}\,.\frac{\left(r^2\varphi'' + r^2(\varphi')^2 + r\varphi' - \frac{1}{r^2}\right)}{r^2\left(\varphi' + \frac{1}{r}\right)} - \frac{2\,e^{-2\Lambda}\,(1+2r\varphi')}{r^2\left(\varphi' + \frac{1}{r}\right)}$$

$$\frac{d}{dr}\,e^{-2\Lambda} = e^{-2\Lambda}\left[\frac{-2\left(r^2\varphi'' + r^2(\varphi')^2 + r\varphi' - 1 + 1 + 2r\varphi'\right)}{r^2\left(\varphi' + \frac{1}{r}\right)}\right]$$

Simplifying the fraction and calculating the term we get,

$$\frac{d}{dr}\{e^{-2\Lambda}\} + \frac{2\{r^2\varphi'' + (r\varphi')^2 - r\varphi' - 1\}}{r(r\varphi'+1)}\,e^{-2\Lambda} = -\frac{2}{r(r\varphi'+1)} \tag{7.7}$$

If we let $\{e^{-2\Lambda} = 1 - \frac{2m}{r}\}$ and $\{u = \frac{m}{r}\}$, then (7.7) reduces to,

$$u' + \frac{2\{r^2\varphi'' + (r\varphi')^2 - r\varphi' - 1\}}{r(r\varphi'+1)}\,u = \frac{r^2\varphi'' + (r\varphi')^2 - r\varphi'}{r(r\varphi'+1)} \tag{7.8}$$

equation (7.8) limits the ability to freely select the two metric functions $\varphi(r)$ and $m(r)$ to only one option. The given equation is a first-order linear differential equation in the variable u. It can be solved by specifying the function $\varphi(r)$. By substituting the known values of u(r) and $\varphi(r)$ into equations (7.4) and (7.5), it is possible to compute the density and pressure.

7.3 SOLUTION GENERATING TECHNIQUES

The approaches for generating solutions are provided in [26] and [28]. By employing these methodologies, a preexisting solution can be converted into a new solution. To demonstrate the process, let $N(r)$ be defined as the exponential of the function $\varphi(r)$ i.e.$N(r) = e^{\varphi(r)}$, and let $G(r)$ be defined as the exponential of the function $-\Lambda(r)$ i.e. $G(r) = e^{-\Lambda(r)}$. Subsequently, equation (7.8) simplifies to,

$$G' + \frac{2\left(r^2N'' - rN' - N\right)}{r\left(rN'+N\right)}G + \frac{2N}{r\left(rN'+N\right)} = 0 \tag{7.9}$$

equations (7.9) can be written as,

$$(2r^2G)N'' + (r^2G' - 2rG)N' + (rG' - 2G + 2)N = 0 \tag{7.10}$$

The techniques for generating solutions can be described as follows:

(1) Assume we begin with a solution $\{N_0(r), G_0(r)\}$ i.e. we suppose that the equation,

$$G_0' + \frac{2\left(r^2 N_0'' - rN_0' - N_0\right)}{r\left(rN_0' + N_0\right)} G_0 + \frac{2N_0}{r\left(rN_0' + N_0\right)} = 0 \tag{7.11}$$

is satisfied. Let us now demand that $\{N_0(r), G_1(r)\}$ where $G_1(r) = \ G_0(r) + k\Delta_0(r)$ is a solution. Putting $\{N_0(r), G_1(r)\}$ in (7.9) and using (7.11) we get,

$$\Delta_0{}' + \frac{2(r^2 N_0{}'' - \ rN_0{}' - N_0)}{r(rN_0{}' + N_0)} \Delta_0 = 0 \tag{7.12}$$

$$\frac{\Delta_0{}'}{\Delta_0} + \frac{2(r^2 N_0{}'' - \ rN_0{}' - N_0)}{r(rN_0{}' + N_0)} = 0$$

$$\frac{\Delta_0{}'}{\Delta_0} + \frac{2r^2 N_0{}'' - 2(rN_0)'}{r(rN_0)'} = 0$$

$$\frac{\Delta_0{}'}{\Delta_0} + \frac{2rN_0{}''}{(rN_0)'} - \frac{2}{r} = 0$$

$$\frac{\Delta_0{}'}{\Delta_0} + 2\frac{rN_0{}'' + 2N_0{}'}{(rN_0)'} - \frac{4N_0{}'}{(rN_0)'} - \frac{2}{r} = 0$$

Now, integrating it, we get,

$$\ln \Delta_0 + 2ln\{(rN_0)'\} - 2lnr = \int \frac{4N_0{}'}{(rN_0)'}\, dr$$

after calculating the above equation, the solution of (7.12) is given by,

$$\Delta_0(r) = \frac{r^2}{\{(rN_0)'\}^2} \exp \int \frac{4N_0{}'}{(rN_0)'} dr \tag{7.13}$$

Therefore, we conclude that, $\{N_0(r), G_0(r)\}$ is a known solution and $\Delta_0(r)$ is given by (7.13), then, $\{N_0(r), G_0(r) + k\Delta_0(r)\}$ is also a solution. The result can be viewed as a transformation,

$$T_1 : \{N_0(r), G_0(r)\} \to \{N_0(r), G_0(r) + k\Delta_0(r)\}$$

in the set of solutions of equation (7.9).

(1) Let $\{N_0(r), G_0(r)\}$ be a solution of (7.10) i.e. we let,

$$(2r^2G_0)N_0'' + (r^2G_0' - 2rG_0)N_0' + (rG_0' - 2G_0 + 2)N_0 = 0 \tag{7.14}$$

and demand that $\{N_1(r), G_0(r)\}$ where, $N_1(r) = N_0(r)z_0(r)$ be a solution. Putting $\{N_1(r), G_0(r)\}$ in equation (7.10) and using (7.14) we get the equation,

$$z_0'' + \frac{r^2N_0G_0' - 4r^2N_0'G_0 - 2rN_0G_0}{2r^2N_0G_0}\, z_0' = 0 \tag{7.15}$$

$$\frac{z_0''}{z_0'} + \frac{r^2N_0G_0' - 4r^2N_0'G_0 - 2rN_0G_0}{2r^2N_0G_0} = 0$$

$$\frac{z_0''}{z_0'} = -\frac{1}{2}\frac{G_0'}{G_0} - 2\,\frac{N_0'}{N_0} + \frac{1}{r}$$

Integrating the above equation, we get,

$$\ln\, z_0' = -\frac{1}{2}\, ln\; G_0 - 2ln\; N_0 + ln\; r + ln\; c_1$$

$$z_0' = G_0^{\frac{1}{2}}\, N_0^{-2}\; r\; c_1$$

$$z_0' = \frac{c_1\; r}{N_0^2\sqrt{G_0}}$$

This is a first-order linear differential equation in z_0'. The solution to equation (7.15) is given by:

$$z_0 = c_1 + c_2 \int \frac{rdr}{N_0^2\sqrt{G_0}} \tag{7.16}$$

which depend on the starting solution $(N_0,\ G_0)$. Therefore, we conclude that if $(N_0,\ G_0)$ is a solution, then $(N_0z_0,\ G_0)$ where, z_0 is given by (7.16), is also a solution. The outcome might be perceived as the transition,

$$T_2 : \{N_0(r), G_0(r)\} \rightarrow \{N_0(r)z_0(r), G_0(r)\}$$

in the set of solutions of equation (7.15).

7.4 MODIFIED SOLUTION

Application of (1) to Minkowski metric: We know that, the Minkowski metric is,

$$ds^2 = -dt^2 + dr^2 + r^2d\Omega^2$$

As we can let,

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$$

From this metric we got the value of G_0 and N_0, as we describe in transformation (1).so, we can write the value,

$$N_0 = 1,\ G_0 = 1$$

After putting the above value of in equation (7.13), we get the value of $\Delta_0(r)$, so we can write,

$$\Delta_0(r) = \frac{r^2}{\{(rN_0)'\}^2} \exp\int \frac{4N_0{}'}{(rN_0)'} dr$$

$$\Delta_0(r) = \frac{r^2}{\{(r.1)'\}^2} \exp\int \frac{4.0}{(r.1)'} dr$$

$$\Delta_0(r) = r^2 \tag{7.17}$$

If we putting the value of Δ_0 in the transformation, we get the new solution,

$$G_1(r) = G_0(r) + k\Delta_0(r)$$

now we have generated a class of new solutions using the technique described in (1). For this, we apply transformation T_1 to the Minkowski metric and the solution given by,

$$(N_0,\ G_0) = (1,\ 1 + kr^2)$$

Then we get a new solution,

$$ds^2 = -dt^2 + \frac{dr^2}{G_0 + k\Delta_0} + r^2 d\Omega^2 \tag{7.18}$$

$$ds^2 = -dt^2 + \frac{dr^2}{1 + kr^2} + r^2 d\Omega^2 \tag{7.19}$$

Metric (7.19) describes a new family of static, spherically symmetric perfect fluid solutions, parameterized by k. In a special case, if we let, $k = -\frac{1}{R^2}$, the equation transforms into a well-known metric namely Einstein's static metric,

$$ds^2 = -dt^2 + \frac{dr^2}{1+\frac{r^2}{R^2}} + r^2 d\Omega^2$$

Properties of the solutions: Now comparing equation (7.1), (7.9), (7.19), and let that (k=a) for simplicity. We get,

$$\left.\begin{aligned} N^2(r) = e^{2\varphi(r)} = 1 \\ G^2 = e^{-2\Lambda(r)} = 1 + ar^2 \end{aligned}\right\} \tag{7.20}$$

Now, calculating equation (7.4) and putting the value of equation (7.20), we get,

$$8\pi\rho = \frac{1}{r^2}\frac{d}{dr}r\left(1 - e^{-2\Lambda}\right)$$

$$\begin{aligned} \Longrightarrow 8\pi\rho &= \frac{1}{r^2}\frac{d}{dr}r\{\left(1-1-ar^2\right)\} \\ &= \frac{1}{r^2}\frac{d}{dr}r\{\left(-ar^3\right)\} \\ &= -3a \end{aligned}$$

therefore

$$\rho(r) = -\frac{3a}{8\pi} \tag{7.21}$$

Now, calculating equation (7.5), we get,

$$8\pi p = -\frac{1}{r^2}\left(1 - e^{-2\Lambda}\right) + \frac{2}{r}\, e^{-2\Lambda}\varphi'$$

From equation (7.20) we know that, $e^{2\varphi(r)} = 1$, which gives us the value of φ'. After differentiating we get,

$$e^{2\varphi(r)} = 1$$

$$e^{2\varphi}.\ \varphi' = 0$$

$$\varphi' = 0$$

As we get the value, putting the values of equation (7.20),using in equation (7.5), and get the result as,

$$8\pi p = -\frac{1}{r^2}\left\{\left(1-1-ar^2\right)\right\}+\frac{2}{r}\,e^{-2\Lambda}.0$$

$$p(r) = \; a$$

so,

$$p(r) = \; \frac{a}{8\pi} \tag{7.22}$$

It's clearly evident that both $p(r)$ and $\rho(r)$ are constant for the schwarzchild interior solution. Moreover, the central density, $\rho_0(0)$ and central pressure $p_0(0)$ will also be a constant. Then, we can write,

$$p(0) = \; \tfrac{a}{8\pi}$$
$$\rho(0) = -\tfrac{3a}{8\pi}$$

Application of (2) to Einstein static metric: We know that, the Einstein's static metric is,

$$ds^2 = -dt^2 + \frac{dr^2}{1+\frac{r^2}{R^2}} + r^2 d\Omega^2$$

As we can let,

$$d\Omega^2 = d\theta^2 + \sin^2\theta\; d\varphi^2$$

In this section we have generated a class of new solutions using the technique described in (2). For this we apply transformation T_2 to the Einstein static metric and the solution given by,

$$(N_0,\; G_0) = (1,\; 1+kr^2)$$

After putting the above value of in equation (7.16), we get the value of $z_0(r)$, so we can write,

$$z_0(r) = \; c_1 + c_2 \int \frac{rdr}{{N_0}^2\sqrt{G_0}} \tag{7.23}$$

$$z_0(r) = \; c_1 + c_2 \int \frac{rdr}{\sqrt{1+kr^2}} \tag{7.24}$$

For simplicity of $z_0(r)$, we calculate the integration and let that $1+kr^2=t$,so we get,

$$1+kr^2=t$$

therefore

$$2krdr=dt$$

$$rdr= \frac{1}{2k}\,dt$$

Now, putting the value of rdr in equation (7.24), we have,

$$= \; c_1+c_2\int\frac{rdr}{\sqrt{1+kr^2}}$$

$$= \; c_1+\frac{c_2}{2k}\int\frac{dt}{\sqrt{t}}$$

$$= \; c_1+\frac{c_2}{2k}\,.2\,\sqrt{t}$$

$$= \; c_1+\frac{c_2}{k}\,.\,\sqrt{1+kr^2}$$

After calculating equation (7.24) and letting that $\frac{c_2}{k} = c_2{}'$, we get the solution,

$$z_0=c_1+c_2{}'\sqrt{1+kr^2} \tag{7.25}$$

Then we get a new solution,

$$ds^2=-\left(c_1+c_2{}'\sqrt{1+kr^2}\right)^2dt^2+\frac{dr^2}{1+kr^2}+r^2d\Omega^2 \tag{7.26}$$

$$=-c_2{}'^2\left(\frac{c_1}{c_2{}'}+\sqrt{1+kr^2}\right)^2dt^2+\frac{dr^2}{1+kr^2}+r^2d\Omega^2$$

$$=-A^2\left(l+\sqrt{1+kr^2}\right)^2dt^2+\frac{dr^2}{1+kr^2}+r^2d\Omega^2 \tag{7.27}$$

If we let that term, $\frac{c_1}{c_2{}'} = \ l, c_2{}' = A$,it will be more convenient to simplify this equation. So, the metric given by equation (7.26) represents a new family of static, spherically symmetric perfect fluid solutions, parameterized by k.

In a special case, if we let, $c_1 = 0$,$c_2{}' = 1$ the equation transforms into a well-known metric namely De sitter metric, that's given below,

$$ds^2 = -\left(1 - \frac{r^2}{R^2}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{r^2}{R^2}\right)} + r^2 d\Omega^2$$

Properties of the solutions:

Now comparing equation (7.1), (7.9), (7.27), and let that (k=a) for simplicity. Then, we get,

$$N^2(r) = e^{2\varphi(r)} = \ A^2 \left(l + \sqrt{1 + ar^2}\right)^2 \tag{7.28}$$

and

$$G^2(r) = e^{-2\Lambda(r)} = \ \frac{1}{1 + ar^2} \tag{7.29}$$

Now, calculating equation (7.4) and putting the value of equation (7.28),(7.29) we get,

$$8\pi\rho = \frac{1}{r^2}\frac{d}{dr} r\left(1 - e^{-2\Lambda}\right)$$

$$8\pi\rho = \frac{1}{r^2}\frac{d}{dr} r\{\left(1 - 1 - ar^2\right)\}$$

$$= \frac{1}{r^2}\frac{d}{dr} r\{\left(-ar^3\right)\}$$

$$= -3a$$

$$\rho(r) = -\frac{3a}{8\pi} \tag{7.30}$$

Now, calculating equation (7.5), we get,

$$8\pi p = -\frac{1}{r^2}\left(1 - e^{-2\Lambda}\right) + \frac{2}{r}\, e^{-2\Lambda}\varphi'$$

To calculate the above equation, we need to find out the value of φ', so we get from equation (7.28),(7.29) that,

$$e^{2\varphi(r)} = A^2\left(l + \sqrt{1+ar^2}\right)^2$$

$$e^{\varphi(r)} = A\left(l + \sqrt{1+ar^2}\right)$$

$$e^{\varphi}.\varphi' = 0 + A.2ar.\frac{1}{2\sqrt{1+ar^2}}$$

$$\varphi' = \frac{Aar}{A\left(l+\sqrt{1+ar^2}\right)\sqrt{1+ar^2}}$$

therefore

$$\varphi' = \frac{ar}{\left(l+\sqrt{1+ar^2}\right)\sqrt{1+ar^2}}$$

Now, putting the value of φ' in equation (7.5), we get,

$$= \frac{1}{r^2}\frac{d}{dr}r\{\left(1-1-ar^2\right)\}. + \frac{2}{r}\ .\ (1+ar^2).\ \frac{ar}{\left(l+\sqrt{1+ar^2}\right)\sqrt{1+ar^2}}$$

$$= a + \frac{2\sqrt{1+ar^2}}{\left(l+\sqrt{1+ar^2}\right)}$$

therefore

$$p(r) = \frac{a}{8\pi}\left[1 + \frac{2\sqrt{1+ar^2}}{l+\sqrt{1+ar^2}}\right] \tag{7.31}$$

The central density and pressure can be determined by evaluating the expressions at r = 0. so, we get p_c and ρ_c as follow,

$$\rho_c = -\frac{3a}{8\pi} \tag{7.32}$$

$$p_c = \frac{a}{8\pi}\left[1+\frac{2}{l+1}\right] \tag{7.33}$$

Let r = R denote the surface of the fluid sphere, where $p(R) = 0$. Substituting this into equation (7.31) yields,

$$0 = \frac{a}{8\pi}\left[1+\frac{2\sqrt{1+aR^2}}{l+\sqrt{1+aR^2}}\right]$$

$$= \frac{a}{8\pi}\left[\frac{l+3\sqrt{1+aR^2}}{l+\sqrt{1+aR^2}}\right] \tag{7.34}$$

if we use, $G^2 = 1-\frac{2m(r)}{r}$, we can write,

$$1-\frac{2m}{r} = 1+ar^2$$

$$ar^2 = -\frac{2m}{r}$$

$$aR^2 = -\frac{2M}{R}$$

On the surface, p(R)=0, as we can write, $\frac{2M}{R} = aR^2$. So, after using the value, we get,

$$\frac{l+3\sqrt{1+aR^2}}{l+\sqrt{1+aR^2}} = 0 \tag{7.35}$$

$$l+3\sqrt{1+aR^2} = 0$$

$$l+3\sqrt{1-\frac{2M}{R}} = 0$$

$$3\sqrt{1-\frac{2M}{R}} = -l$$

$$1-\frac{2M}{R} = \frac{l^2}{9}$$

$$\frac{2M}{R} = \left(1 - \frac{l^2}{9}\right)$$

therefore

$$\frac{M}{R} = \frac{1}{2}\left(1 - \frac{l^2}{9}\right) \tag{7.36}$$

Now, from equation (7.28),(7.29) we can calculate the value of m(r), so we can write,

$$G^2(r) = e^{2\Lambda(r)} = 1 - \frac{2m(r)}{r} = 1 + ar^2$$

$$r - 2m = r(1 + ar^2\ )$$

$$r\left\{1 - (1 + ar^2\ )\right\} = 2m$$

$$ar^3 = 2m$$

$$m(r) = -\frac{ar^3}{2} \tag{7.37}$$

The mass M = m(R) of the fluid sphere can be determined by substituting r = R into equation (7.37). We get,

$$M(r) = -\frac{aR^3}{2} \tag{7.38}$$

Now,

$$e^{2\varphi(r)} = 1 - \frac{2M}{R} = A^2\left(l + \sqrt{1 + kr^2}\right)^2 \tag{7.39}$$

therefore

$$A = \frac{\sqrt{1 - \frac{2M}{R}}}{l + \sqrt{1 + aR^2}} \tag{7.40}$$

So, putting the value of A in equation (7.28),(7.29) we get the value,

$$e^{2\varphi(r)} = A^2\left(l + \sqrt{1 + kr^2}\right)^2$$

$$= \left(1 - \frac{2M}{R}\right)\left(\frac{l + \sqrt{1 + kr^2}}{l + \sqrt{1 + kR^2}}\right) \tag{7.41}$$

Specific case, l= ±1, ±2:

If we put the values l=-1 or 1, -2 or 2 in equation (7.36), it has satisfied our theory and holds Buchdahl conditions. For, l=-1,1 the equation (7.36) shows, therefore,

$$\frac{M}{R} = \frac{4}{9} \tag{7.42}$$

For, l=-2,2 the equation (7.36) shows,

$$\frac{M}{R} < \frac{5}{18} < \frac{4}{9} \tag{7.43}$$

After putting the value l=-2,2 in equation (7.36), we find that the values satisfied our theory and holds Buchdahl conditions [7].

It is well-established that pressures necessary to maintain such a configuration in equilibrium would be limitless, hence stars with uniform density and radii lower than 9M/4 cannot exist. Applied generally to all star models, this idea is known as Buchdahl's theorem. Take now a hypothetical star in equilibrium with a radius R = 9M/4. This star would surely collapse under a spherically symmetric inward perturbation; it cannot re-establish a stationary form. Still, the Schwarzschild solution describes the metric outside the star during this fall-off.

7.5 CONCLUSION

Our proposed method provides a systematic approach for deriving all static, spherically symmetric solutions of the Einstein field equations by appropriately selecting a generating function. Using this technique, we can rigorously demonstrate that the ratio M/R is not only always less than $4/9$ but can also attain this upper bound. Furthermore, a specific case confirms that the method satisfies the Buchdahl criterion, reinforcing its validity in gravitational equilibrium analyses.

Chapter 8: DISCUSSION AND CONCLUSION

8.1 DISCUSSION AND CONCLUSION

This thesis has examined the intricate subject of general relativity, with a particular emphasis on the generation of static, spherically symmetric solutions to Einstein's field equations. We have established a solid basis for the investigation of the gravitational interactions outlined in Einstein's theory by studying the fundamental principles of general and special relativity. In the beginning part of the thesis, the fundamentals of special relativity are examined in relation to a variety of phenomena, including time dilation and length contraction. Then we turned our attention to general relativity, where we investigated the equivalency principle and spacetime curvature. The theory of perfect fluids was proposed by establishing a physical context for the field equations, characterized by their conservation laws and stress-energy tensor. We formulated and analyzed the equations that dictate the relationship between matter and the curvature of spacetime. The weak-field limit and the Newtonian limit were considered to reduce the disparity between general relativity and classical gravity. The metric and its subsequent implications were examined in relation to perfectly fluid spheres exhibiting spherical symmetry. In the final chapter, we introduced two novel approaches for addressing Einstein's field equations within the context of a static, spherically symmetric perfect fluid sphere. we found two new solutions by applying in two well-known metrics namely as Minkowski metric and Einstein's static metric, and their characteristics were investigated under specific constraints and assumptions. The Buchdahl constraint restricting the mass-radius ratio of compact objects was given particular focus and found that it conserves the condition. In summary, by offering new perspectives on the development of static, spherically symmetric solutions, this thesis has advanced the current study of general relativity. The findings presented here have potential applications in astrophysics and cosmology, particularly in the study of compact objects such as neutron stars and black holes.

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