

DESIGNING CONTROL SCHEMS FOR CHAOTIC SYSTEMS VIA SLIDING MODE CONTROL



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Declaration

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Dedicated To My Family Members

List of Publications

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Approval/Declaration by the Supervisor

This is to certify that Afsana Afroja has carried out this research work under my supervision, and that she has fulfilled the relevant Academic ordinance of the Chittagong University of Engineering and Technology, so that she is qualified to submit the following Thesis in the application for the degree of MASTER of PHILOSOPHY in MATHEMATICS. Furthermore, the Thesis complies with the PLAGIARISM and ACADEMIC INTEGRITY regulation of CUET.

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Abstract

Among various robust control methods, Sliding Mode Control (SMC) has attracted considerable interest in theoretical research due to its unique features. SMC is well known for its robustness to matched and bounded uncertainties, reduction in the order of the system during the sliding phase, simplified decoupling of system dynamics, and its ability to achieve zero steady-state error. These characteristics make SMC a powerful tool in designing control systems for uncertain and nonlinear environments.

This thesis presents a comprehensive study on the control and synchronization of chaotic systems, focusing on the Modified Lorenz System (MLS) and the Liu financial dynamical system. Chaotic behavior, with its sensitivity to initial conditions, presents major challenges in nonlinear control. Sliding Mode Control (SMC) is applied to coupled MLS to ensure robust synchronization through a designed sliding surface and control law, with convergence proven via Lyapunov stability and validated through numerical simulations. In the Liu financial system, both SMC and Passive Control (PC) are implemented and compared. While SMC provides rapid synchronization (within $t \geq 1$, error reduced from 5.67 to 0.02), PC achieves synchronization more gradually (within $t \leq 13$, error reduced from 6.21 to 0.03) using a simpler, single-controller strategy. The results highlight key trade-offs between speed, robustness, and implementation complexity, offering valuable insights into managing chaos in nonlinear and financial systems.

These limitations have motivated ongoing research aimed at improving SMC design. The core challenge remains how to develop a simple yet effective sliding mode control technique that retains its robustness and zero-error tracking while minimizing chattering and relaxing the need for precise uncertainty bounds.

বিমূর্ত

বিভিন্ন রকমের রোবাস্ট কন্ট্রোল পদ্ধতির মধ্যে, স্লাইডিং মোড কন্ট্রোল (SMC) এর অনন্য বৈশিষ্ট্যের কারণে এটি তাত্ত্বিক গবেষণায় ব্যাপক আগ্রহ অর্জন করেছে। স্লাইডিং মোড কন্ট্রোল (SMC) মিলে যাওয়া এবং সীমিত অনিশ্চয়তার প্রতি এর রোবাস্টনেস, স্লাইডিং ধাপে সিস্টেমের অর্ডার হ্রাস, সিস্টেম ডায়নামিক্সকে সহজে ডিসকাপল করার ক্ষমতা, এবং জিরো স্টেডি-স্টেট এরর অর্জনের সক্ষমতার জন্য পরিচিত। এসব বৈশিষ্ট্য (SMC) কে অনিশ্চিত ও ননলিনিয়ার পরিবেশে কন্ট্রোল সিস্টেম ডিজাইনের জন্য একটি শক্তিশালী উপকরণ হিসেবে প্রতিষ্ঠিত করেছে।

এই থিসিসে ক্যাণ্ডটিক সিস্টেমের কন্ট্রোল এবং সিনক্রোনাইজেশন নিয়ে একটি ব্যাপক গবেষণা উপস্থাপন করা হয়েছে, যেখানে ফোকাস করা হয়েছে মডিফায়েড লরেন্স সিস্টেম (MLS) এবং লিউ ফাইন্যান্সিয়াল ডায়নামিক্যাল সিস্টেমের উপর। প্রাথমিক শর্তের প্রতি সংবেদনশীলতার কারণে ক্যাণ্ডটিক আচরণ ননলিনিয়ার কন্ট্রোলে বড় চ্যালেঞ্জ তৈরি করে। স্লাইডিং মোড কন্ট্রোল (SMC) প্রয়োগ করা হয়েছে কাপলড MLS - এ, যেখানে একটি নির্ধারিত স্লাইডিং সারফেস এবং কন্ট্রোল ল এর মাধ্যমে রোবাস্ট সিনক্রোনাইজেশন নিশ্চিত করা হয়েছে, যার কনভার্জেন্স লিয়াপুনভ স্থিতিশীলতার মাধ্যমে প্রমাণিত এবং সংখ্যাাত্মক সিমুলেশন দ্বারা যাচাই করা হয়েছে। লিউ ফাইন্যান্সিয়াল সিস্টেমে (SMC) এবং প্যাসিভ কন্ট্রোল (PC) উভয়ই প্রয়োগ ও তুলনা করা হয়েছে। যেখানে (SMC) দ্রুত সিনক্রোনাইজেশন নিশ্চিত করে ($t \geq 1$ এর মধ্যে, ত্রুটি ৫.৬৭ থেকে ০.০২-এ হ্রাস), সেখানে (PC) ধীরে ধীরে সিনক্রোনাইজেশন অর্জন করে ($t \leq 10$ এর মধ্যে, ত্রুটি ৬.২১ থেকে ০.০৩-এ হ্রাস) একটি সহজ, একক কন্ট্রোলার ভিত্তিক কৌশল ব্যবহার করে। ফলাফলগুলো স্পষ্টভাবে গতির, রোবাস্টনেসের ও বাস্তবায়নের জটিলতার মধ্যে মূল ট্রেড-অফগুলো তুলে ধরে, যা ননলিনিয়ার এবং ফাইন্যান্সিয়াল সিস্টেমে ক্যাণ্ডস নিয়ন্ত্রণে গুরুত্বপূর্ণ অন্তর্দৃষ্টি প্রদান করে।

এসব সীমাবদ্ধতাই (SMC) ডিজাইন উন্নয়নের লক্ষ্যে চলমান গবেষণাকে অনুপ্রাণিত করেছে। মূল চ্যালেঞ্জ হলো, কীভাবে একটি সহজ কিন্তু কার্যকর স্লাইডিং মোড কন্ট্রোল কৌশল তৈরি করা যায়, যা এর রোবাস্টনেস ও জিরো-এরর ট্র্যাকিং বজায় রাখবে, আবার চ্যাটারিং হ্রাস করবে এবং অনিশ্চয়তার সুনির্দিষ্ট সীমার প্রয়োজনে শিথিলতা দেবে।

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Nomenclature

VSS	Variable Structure System.
VSC	Variable Structure Control.
SMC	Sliding Mode Control.
TSM	Terminal Sliding Mode.
TA	Twisting Algorithm.
SMC	Sliding Mode Control.
LTI	Linear Time Invariant.
SOSM	Second Order Sliding Mode.
HOSM	Higher Order Sliding Mode.
FTSM	Fast Terminal Sliding Mode.
DT	Discrete Time.
SISO	Single Input Single Output.
$d(t)$	Time-Varying Disturbance.
$x_d(t)$	Desired State.
u	Single-Valued Control Input.
$u_{eq}(t)$	Equivalent Control.
σ	Sliding Variable.
$S(t)$	Sliding Surface.
δ	Matched Disturbance.
ϕ	Unmatched Disturbance.
H_∞	Optimal Control.
$G(x)$	Projection Matrix.

Chapter 1: INTRODUCTION

1.1 General Overview

Mathematical models of complicated behaviors found in social, engineering, and natural systems are known as nonlinear dynamical systems. Nonlinear systems, in contrast to linear systems, can display a wide range of dynamics, such as bifurcations, periodic oscillations, fixed points, and chaotic behavior. A characteristic of chaos known as sensitivity to initial conditions, these occurrences frequently occur in systems controlled by nonlinear differential equations, where slight modifications to the original conditions can produce radically different results [1].

In physics, biology, engineering, economics, and many other fields, modeling phenomena depends significantly on nonlinear dynamical systems. When a system's governing equations do not produce linear dependent variables and their derivatives, the system is said to be nonlinear. Limit cycles, bifurcations, chaos, and numerous equilibria are some of the phenomena that define these systems. Nonlinear systems are more difficult to analyze mathematically than linear systems, but they describe real-world phenomena more accurately.

Within the field of nonlinear dynamics, chaos theory investigates deterministic systems that seem random because of their extreme sensitivity. Even though these systems are subject to exact mathematical laws, they can evolve in strange and seemingly chaotic ways. Numerous disciplines, including as physics, biology, economics, and engineering, exhibit this chaotic pattern.

In the early 20th century, the idea of chaos in dynamical systems was thoroughly investigated for the first time. The discovery of deterministic chaos in atmospheric models by Edward Lorenz in the 1960s, however, brought it a lot of attention. Lorenz demonstrated the "butterfly effect," which is the inability of even basic deterministic systems to generate highly sensitive and unpredictable behaviors. Since then, several chaotic systems have been discovered, such as the Lü system, Chua's circuit, Rössler system, Lorenz system, and others [2].

Even though chaos is deterministic, it behaves erratically and sporadically. In the control domain, this sensitivity to initial conditions presents both a problem and an opportunity. Although chaos can be harmful, causing biological systems to exhibit inconsistencies or engineering systems to become unstable, it can also be used for positive outcomes, such as secure communication and the creation of random numbers.

Using small perturbations or control inputs, a chaotic system can be guided toward a desired behavior. This technique is known as chaos control. The main objective is to synchronize two or more chaotic systems or stabilize an unstable periodic orbit embedded in a chaotic attractor. Numerous chaos control strategies, including sliding mode control, adaptive control, linear feedback control, and optimal control, have been developed since the groundbreaking work of Ott, Grebogi, and Yorke (OGY method) in 1990[3]. Controlling chaos has emerged as a critical challenge and opportunity in nonlinear system theory. Using feedback mechanisms or outside interventions, chaos control seeks to steer chaotic trajectories toward desirable behaviors or stabilize unstable periodic orbits. Applications ranging from electrical circuits and secure communications to weather forecasting and ecological modeling require efficient control systems.

SMC stands out among them due to its resilience to external shocks and system uncertainty. To provide desired dynamic properties, the system trajectory is pushed to slide over a pre-defined surface (sliding surface) in the state space using this variable structure control method. Because of its intrinsic resilience, SMC works well for synchronization and stability in chaotic systems. Chaos control also includes the synchronization of chaotic systems.

Synchronization is the process of modifying a response system to correspond with the actions of a driving system, as first suggested by Pecora and Carroll in 1990 [4]. Numerous fields, including neural networks, biological systems, and secure communication, can benefit from this. Designing appropriate controllers that maintain synchronization in spite of the chaotic and nonlinear character of the systems involved is the difficult part.

The focus of recent developments is on better control strategies that blend flexibility and resilience. To improve performance in real-world scenarios, researchers have investigated hybrid approaches, data-driven control, and fractional-order systems. An

ongoing research field is the control of hyperchaotic and multi-scroll systems, as well as systems with uncertainty and time delays.

The study of chaos has provided insights into a variety of real-world phenomena, including laser dynamics, stock market swings, cardiac rhythms, and turbulence. It is crucial to comprehend and manage these behaviors to guarantee system stability and the intended performance.

Depending on the individual systems' fundamental characteristics and physical parameter values, nonlinear dynamical systems can display a variety of behavioral patterns. Continuous chemical reactors and continuous polymerization reactors are two dynamical systems that have drawn particular attention because of their complex dynamics. Complex dynamic phenomena include numerous steady-state behaviors, limit cycle oscillations, and chaos are displayed by these nonlinear dynamical systems.

Chaos is derived from the Greek word “Khaos,” which means “gaping void”. Experts in mathematics state that defining chaos can be tricky, but recognizing it is straightforward. In other words, chaos denotes to a state of total uncertainty or estimating future behavior in complex natural phenomena. This mathematical branch is utilized in various fields such as physics, engineering, economics, biology, and philosophical with insights from Morse (1967) [5]. The fundamental point is that even slight modifications to initial parameters —such as rounding errors in numerical methods can result in drastically distinct results for chaotic systems, projecting long-term trends generally unfeasible. One of the basic theories in our existence is chaos theory. It is the study of complex, nonlinear dynamic systems. This area of mathematics examines systems that exhibit chaotic behaviors despite their apparent order (determinism). Numerous academic disciplines have taken notice of chaotic attractors. There are several uses for chaos in both engineering and natural science. The Lorenz system, which was initially presented as first used for meteorological purposes forecasting, is the first chaotic system. Subsequent research revealed that the electrical circuit [6] and chemical reaction model [7] both contain chaotic attractors. According to research, the traditional the Lorenz system, along with the Chen system and the Lü system [8] are all of them members of the generalized Lorenz systems lineage.

Trajectories originating from chaotic systems are mathematically defined by both general boundlessness and local unpredictability [9]. Since a linear system's local

unpredictability means that its solution is unbounded, a chaotic system must be nonlinear [10], meaning that it must be represented by a nonlinear mathematical model. Nonlinear models [11] have the benefit of being able to describe complicated behavioral pattern with a small number regarding the variables and parameters. The nonlinear mathematical model [12] must therefore depict a chaotic system with properties that are dependent on the beginning conditions because of these features. This refers to commonly it is understood that a chaotic system is a nonlinear, deterministic system characterized by certain fundamental characteristics, including bounded aperiodic long-time behavior, predefined motion, the butterfly phenomenon, and system pathways that consistently pass through each designated point within the phase space, as well as the ability to produce significant variations in the system's condition from a tiny modification in the system's initial state. Researchers in nonlinear fields are driven to examine the intricacy of potential engineering applications, including ecological systems [13], applications in mechanical and chemical engineering, cybersecurity communication methods, and the telecommunications sector, lasers and plasma technologies, and system engineering. Mountains, streams, clouds, oceanic flow, air instability, chaotic patterns can be observed in a wide range of fields, including laser physics, biophysics, chemistry, biology, blood circulation through fractal arterial networks, medicine, electronic circuits, astronomy and epidemiology [14].

The application of synchronization and chaos control techniques is necessary to comprehend and utilize the basic dynamics of chaotic systems. This thesis investigates the use of SMC in the creation of control methods for chaotic systems. Chaotic systems exhibit very complicated and unpredictable behaviors and are dynamic systems that are extremely sensitive to beginning conditions. These systems are not predictable due to their deterministic nature. Chaos behaves differently in deterministic systems, even though it generally displays the random-like behavior found in statistical systems. Since it is usually difficult to accurately forecast the future behavior of a chaotic system, chaos may lead to systems becoming unstable, performing worse, or even collapsing, to the extent that chaos is seen as undesirable and frequently needs to be eliminated.

A variety of techniques are used in modern chaos control methods to manage and stabilize chaotic behavior in dynamical systems. Lyapunov control and other nonlinear control strategies are one of the notable methods. Lyapunov control is based on Lyapunov

stability theory, which is concerned with identifying and ensuring the stability of nonlinear systems. A Lyapunov function is used in Lyapunov control to examine and regulate the dynamics of a system. Lyapunov control is widely used in electrical circuits, robotics, chaotic systems, and other technical applications. It works particularly effectively for nonlinear systems [15].

Different strategies that use nonlinear functions to control and stabilize chaotic systems are referred to as nonlinear control methods. This category includes feedback control, sliding mode control, adaptive control and others. Many scholars have been interested in the control of chaotic systems during the last few decades. In the author's novel uncertain chaotic dynamical system (the Liu system), chaos was eliminated through the deployment of a sliding mode controller. To remove chaos from a novel class of chaotic systems, a strong adaptive sliding mode controller is used. Sliding mode control is a simple approach to chaos management that has been effectively applied to control nonlinear systems and uncertain systems. Sliding mode control and passive control are two of the most modern ways to accomplish chaos synchronization in linked or forced chaotic systems, however many other ideas have also been put forth.[16]

The backstepping approach is used in [17] to manage chaos in the Lorenz, Chen, and Lu systems. In the Newton-Leipnik system, linear feedback controllers are used to regulate chaos. In the theory and applications of fractional calculus, numerous intriguing advancements and findings have been found in the literature.

Interacting chaotic systems can exhibit several synchronization regimes, including anticipatory, phase, lag, generalized, and overall synchronization [18].

1.2 Objectives with Specific Aims and Possible Outcome

The primary causes of the issues with traditional SMC techniques are their highly nonlinear behaviors and their imprecise understanding of complex systems. Therefore, to address the negative impacts of uncertainties and nonlinearities on dynamic systems, sophisticated control strategies are desperately needed. Many researchers concentrate on developing advanced SMC techniques and SMC-based methods for complex, time-varying dynamic models, as SMC provides an optimal framework that ensures strong robustness and

guarantees the asymptotic stability of the closed-loop system. Despite the variety of ways attempted to solve this issue, no ideal answer has been found. In fact, most control solutions created for complex systems necessitate an understanding of the dynamics of uncertain systems beforehand. This is frequently not feasible in practice since it is impossible to obtain information about the dynamics of uncertain systems. As a result, large-scale systems might not be able to use such control methods. Undoubtedly, the creation of a robust controller that can handle complex systems in real time without depending on data about system uncertainties and disturbances remains a top priority.

The main goal of this research is-

- To innovate control schemes using SMC principles for effective stabilization and control of chaotic dynamics
- To design a controller is under systems uncertainty and disturbance torque.
- To the system robustness and steady accuracy.
- To solve the nonlinear tracking problem with continuously switching topologies.
- To drive the systems to slide on a surface, defined in the phase space which is representing the systems target behavior.
- Demonstrate the numerical results using MATLAB or Simulink software for clarifying the effectiveness of control law.

1.3Thesis Outlines

This thesis investigates the design of SMC techniques for a range of complex systems, highlighting their high resilience to uncertainties and external disturbances. The thesis's remaining sections are arranged as follows:

Chapter 1 Describes the theoretical background, objectives with specific aims and possible outcome. We also present the thesis outline of the research problem.

Chapter 2 Offers a concise overview of existing SMC techniques and their evolution. It highlights significant developments in the field, with particular emphasis on controller design methodologies, robustness evaluation, and the main challenges associated with traditional SMC theory and its practical applications.

Chapter 3 Introduces the foundational investigation of the proposed SMC algorithms, serving as the groundwork for the subsequent chapters. It provides a detailed discussion on convergence analysis, supported by rigorous mathematical proofs.

Chapter 4 This chapter studies chaos control in the Liu financial system using Sliding Mode Control (SMC) and Passive Control (PC). It introduces the system, analyzes its chaotic behavior, and designs both control strategies. The part of this chapter (Afroja A, Akter MT and Azam SMS, is published in *Frontiers in Applied Mathematics and Statistics*, 2025, Vol.11.)

Chapter 5 Explores the regulation and synchronization of chaos in Modified Lorenz Systems using sliding mode control. It presents development of control strategies schemes aimed at stabilizing chaotic behavior and achieving synchronization between systems, demonstrating the effectiveness of the proposed methods through theoretical analysis and simulation results.

Chapter 6 Provides the conclusion of the research work. It summarizes the key findings and contributions presented in the previous chapters, highlighting design of control approaches techniques for chaos synchronization. The chapter also outlines the limitations of the current study and suggests potential directions for future research.

Chapter 2: LITERATURE REVIEW

2.1 Introduction

One of the distinctive types of variable structure systems is control of SMC. In a wide range of engineering applications sliding-mode control has demonstrated its effectiveness as a practical and unique resilient control strategy method during the past 20 years. Dynamic system trajectories can be guided towards the predefined sliding surface based on the suggested switching surface and discontinuous controller. It is possible to attain the performance in question. The robustness of restraining system uncertainties and the lowering the order of the system's differential equations from the suggested discontinuous control functions are SMC's two primary benefits. To force the motion paths of dynamic systems to follow the sliding surface over the fixed sliding manifold, a non-smooth control strategy and switching surface design are used. The sliding surface-based control method has been effectively used to bring chaotic systems into synchronization.

For regulation of nonlinear systems with uncertainties, a control technique based on SMC is widely employed because of its intrinsic robustness and finite time convergence. It offers model order reduction in addition to robustness against parametric variations and outside disruptions. Numerous dynamic systems, such as electric drives, power converters, power systems, airplanes, self-driving cars, a variety of mechanical systems, and industrial operations, have used it. The robustness of SMC is due to its discontinuous control law. Nevertheless, it also has the so-called chattering issue. A significant portion of SMC research focuses on the chattering phenomena [19], which gives rise to chattering free SMC, adaptive SMC, and higher order SMC.

The theory of "Variable Structure Systems (VSS)"[20] was first put forth by the Russian scientists and was derived from mathematical work of Philippov in the early 1950s. Since given that high-speed control devices have been invented everywhere, this hypothesis has been expanded. "Stability" and "Consistent performance" against "bounded external disturbances" are issues that this systematic solution addresses. SMC refers to more operational programs related to VSS. This approach uses "highspeed nonlinear control and nonlinear feedback" to switch intermittently on a specified absorbent sliding surface in a short period of time.

Around this sliding surface, SMC produces a limited switching and a discontinuous control law. Consequently, this sliding surface must unavoidably be the direction of the state variables' motion and the dynamics of the system. The "chattering phenomenon,"[21] which should be lessened or removed in some way, is an undesirable issue that arises when such a control law is present and affects the sliding's control. The literature has proposed a few methods, most of which are predicated on the continuous approximation of the discontinuous sign function.

2.2 Sliding Mode Control Framework

Controlling in the presence of constrained external disturbances and perturbation uncertainty is one of the most important aspects of controller design. For this reason, numerous studies on systems with these conditions have been conducted in recent years. The authors of these comprehensive investigations have created reliable control strategies for various kinds of circumstances, which of these techniques, SMC is the most crucial. One of the control strategies for the VCS [22] is the foundation of sliding mode controllers.

2.2.1. Classical Sliding Mode Control Method (CSMC)

There are numerous notes and real-world applications in CSMC. It is because of the sliding mode's appealing characteristics that it is both robust to "perturbation-model uncertainties" and "bounded external disturbances." It additionally offers a "fast-convergence characteristic for a nonlinear system by following a reduced order system" [23]. However, there are two issues with this approach: the "input chattering problem" (chattering phenomena) and the "reaching phase" issue. In addition to these problems, the CSMC controller becomes much more conservative in its use than other design methods. Because the trajectories of motion are obtained by sliding mode dynamics, they cannot have dynamics as the main system [24]. In other words, the following factors cause the unwanted problem of "chattering phenomenon":

- 1)Not being resilient throughout the reaching stage.
- 2) Stimulate high frequency oscillations in the system's sensors or actuators.
- 3) Making use of the controller's big gain.
- 4) Switching of input control signals.

Other issues include the inability to guarantee asymptotic stability, which makes it difficult for the system dynamics to converge to an equilibrium point in a short period of time.

A sliding mode controller's design consists of:

- 1) Choose a "sliding surface" that is robust during the reaching phase and reflects the "desired stable dynamics."
- 2) Explains the "control law" that ensures both "the sliding condition" and "the reaching condition."

Two sections can be used to describe a CSMC's "phase trajectory": the "sliding phase" and the "reaching phase," which occurs when the state trajectory is led towards the sliding surface.

phase" (when the state trajectory is traveling down the sliding surface in the direction of the origin), which will be examined in the next two control modes (Fig. 2.1):

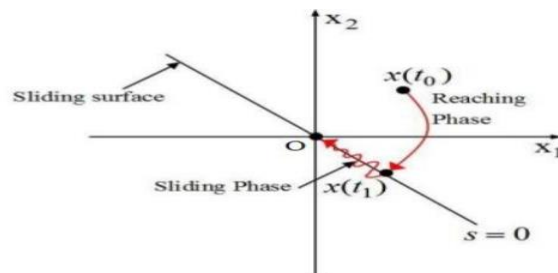


Fig. 2.1: The CSMC in phase plane [25]

2.2.2. Convergence Phase to the Sliding Surface

The motion routes' "reaching phase" from the intended starting point $x(t_0)$ to the sliding surface in a constrained time, or point $x(t_1)$, was clearly shown. The sliding surface $x(t_1)$ is where the moving routes that begin with a start condition outside of it $x(t_0)$ are dragged. Dragged to the sliding surface ($x(t_1)$) are the moving paths that begin with a start condition provided outside the sliding surface ($x(t_0)$). This is referred to as the "approaching phase" or "initial convergence phase". During this stage of the phase trajectory, the SMC system is sensitive to model uncertainties and external disturbances that are unknown but bounded.

The designer should be capable of minimizing or eliminating this sensitivity through various techniques. A key point to note is that when the initial error states are significantly distant from the sliding surface, the discontinuous control gain $K(x(t), t)$ tends to escalate quickly. This rise is attributed to the large magnitude of $s|0|$, rather than being caused by system perturbations or external disturbances. This is the reason the discontinuous control gain $K(x(t), t)$ becomes excessively large. The increasing discontinuous control gain $K(x(t), t)$ compels the error states to reach the sliding surface. This is why some researchers advocate for removing the "reaching phase" altogether.

Up to now, several methods have been introduced to eliminate the reaching phase.

- 1) Suggest an integral sliding mode.
- 2) Utilizing a time-varying sliding surface (or switching sliding surface) [26].
- 3) Implementing a stronger control law [27].
- 4) Discontinuous high-gain feedback control to accelerate the reaching phase [27].

2.2.3. Sliding Mode Phase

Park et al. [20] suggested that the system states should remain on the sliding surface; in other words, the "sliding surface" should be transformed into an "absorbing surface" to facilitate the design of the "appropriate control law." This is also one of the challenges. As a result, the sliding mode controller is quite restrictive when used alongside other controller design methods. This is because the state trajectory of the CSMC system is governed by sliding mode dynamics, which cannot match the original system's [28] dynamics in terms of order.

Once the first stage (designing the absorbent sliding surface) is completed, it transitions into the second stage, which involves determining the "switching control law." This law ensures that the system states remain on the surface once it reaches the sliding mode, thereby maintaining system stability. To achieve this, the control law must be designed so that the vectors of the state variables and their dynamics are tangent to the trajectories leading to the surface.

Mendes observed that in practice, variable structure control (particularly SMC) presents an issue: an "ideal sliding mode" only exists when the state trajectory moves toward

the sliding surface without any chattering, stays on it, and eventually reaches the equilibrium point [27].

A continuous control law $u_{eq}(t)$ that approaches the discontinuous law $u_{sw}(t)$ operating near the switching surface region can be employed to remedy this issue, resulting in a reference tracking with regulated precision, according to Mendes. There is a trade-off between "the accuracy in reference tracking" and "the intensity of chattering in the control input"; the more precisely references are tracked, the more chattering occurs (Fig. 2.2).

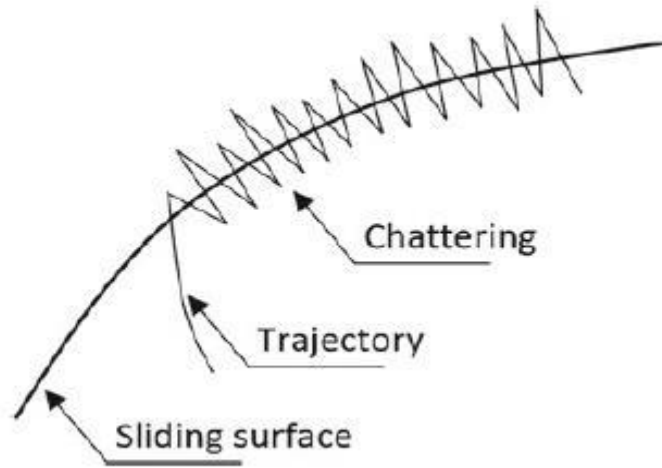


Fig. 2.2: A diagram illustrating the intensity of chattering in the control signal near the sliding surface [29].

Sliding happens along a neighborhood of size equal to 2ϕ (twice the width of the boundary layer) of the sliding surface, as demonstrated by the "boundary layer around the sliding surface" concept (Fig. 2.3):

$$\{x \mid |s| \leq \phi, \quad 0 < \phi\}. \quad (2.1)$$

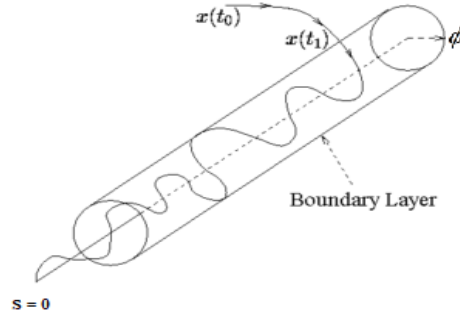


Fig. 2.3: The boundary layer envelops the sliding surface [27]

The system's response states when the boundary layer is implemented around the linear sliding manifold ($s = 0$) is observed in Fig. 2.3. This surface is surrounded by a radius that has been observed to have a set boundary layer thickness (ϕ). After arriving at this sliding surface, the system states continue their erratic sliding motion within this cylinder to the radius ϕ until they converge to the point of equilibrium on $s = 0$.

We examine a nonlinear dynamic system of second order that has Single-Input Single-Output (SISO):

$$\ddot{x}(t) = f(X(t), t) + g(X(t), t) \cdot u(t) + d(t). \quad (2.2)$$

where $d(t)$ is a time-varying disturbance with a known upper bound, $u(t)$ is the control rule, and $x(t)$ is the state vector. The functions $f(X(t), t)$ and $g(X(t), t)$ are typically nonlinear with respect to time and state variables. Their approximations, $f(X(t), t) \rightarrow \hat{f}(X(t), t)$ and $g(X(t), t) \rightarrow \hat{g}(X(t), t)$ are used to ensure that $\dot{s}(t) = 0$ is achieved.

The discontinuous control law is therefore as follows [30]:

$$u(t) = u_{eq}(t) + u_{sw}(t) \quad (2.3)$$

$$u_{eq}(t) = \frac{1}{\hat{g}(X(t), t)} (-\hat{f}(X(t), t) - d(t) + \ddot{x}_d(t) - \lambda \dot{e})$$

$$u_{sw}(t) = -K(X(t), t) \cdot \text{sign}(s(t)).$$

The sliding surface in equation (2.3) has been covered up and down by the discontinuous sign function ($\text{sign}(s)$). The system's bandwidth is represented by λ , a strictly positive constant. The tracking error in the output state is represented by $e(t) = x(t) - x_d(t)$ the tracking derivative error in the output state by $\dot{e}(t) = \dot{x}(t) - \dot{x}_d(t)$ and the desired state

is represented by $x_d(t)$. The discontinuous control gain is the coefficient of the discontinuous sign function $(K(X(t), t))$.

$$K(X(t), t) = F(X(t), t) + \eta. \quad (2.4)$$

It is assumed that the estimation error on $f(X(t), t)$ is bounded by a known function, $F = F(X(t), t)$, so that:

$$|\hat{f}(X(t), t) - f(X(t), t)| \leq F(X(t), t). \quad (2.5)$$

where η is a constant that is strictly positive.

2.3 Lyapunov Stability Theory

In control engineering and systems theory, the stability theory is essential. There are various types of stability, including periodic orbit stability and input-output stability. Specifically, stabilization of steady states is typically defined in terms of Lyapunov, a Russian engineer and mathematician who established the concept [31]. Asymptotic stability, necessary criteria for stability, and other concepts are provided by Lyapunov stability theorems.

Conditions for the linear stability, such as Routh's stability criterion or Nyquist stability criterion, cannot be adapted and applied to the systems for stability analysis because nonlinearities and potential time-varying parameters exist in nonlinear systems.

2.3.1 Equilibrium Point Stability

We consider a system described by the following equations

$$\dot{x} = f(x, t). \quad (2.6)$$

Here, $x \in \mathbb{R}^n$ denotes the vector of state variables, with n being the dimension of the system. The function $f(x, t) \in \mathbb{R}^n$ consists of expressions that depend on the state $x(t)$ and time t .

Let $x_e \in \mathbb{R}^n$ denote an equilibrium point of system (2.1), such that $f(x_e, t) = 0$. For simplicity, and without loss of generality, we consider the equilibrium point to be at the origin, $x_e = 0$, throughout the subsequent definitions and theorems.

Definition 1: At the origin, system (2.6) reaches equilibrium.

- Stability is ensured provided that for any positive number $\epsilon > 0$, one can find a corresponding $\delta > 0$ such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall t \geq 0 \quad (2.7)$$

- If no such δ exists for a given ϵ , the system is deemed unstable
- A system is asymptotically stable provided it is stable and there exists a value of δ such that

$$\|x(0)\| < \delta = \lim_{t \rightarrow \infty} x(t) = 0. \quad (2.8)$$

It is worth noting that Definition (1) pertains to local stability, which characterizes the system's behavior only in the vicinity of an equilibrium point. Therefore, its practical applicability is limited. To overcome this limitation and achieve global stability, the Lyapunov direct method is introduced below.

2.3.2 Lyapunov's Direct Method

Also referred to as the second method of Lyapunov, the direct method provides a means of assessing system stability without explicitly solving the differential equation in (2.6). Lyapunov functions come from theoretical mechanics, where in stable systems, energy acts like a positive value that gets smaller as time goes on. Using this idea, we can create a general energy-like scalar function, called a Lyapunov function, to check the stability of nonlinear systems.

Theorem 1: Let $D \subset R^n$ be a region that includes the origin of the system, and let $V(x): D \rightarrow R$ be a continuously differentiable function such that

$$V(0) = 0 \text{ and } V(x) > 0, \forall x \neq 0. \quad (2.9)$$

and

$$\dot{V}(x) \leq 0. \quad (2.10)$$

Then the equilibrium point $x = 0$ is stable. Furthermore, if

$$\dot{V}(x) < 0, \forall x \neq 0. \quad (2.11)$$

Then the equilibrium at $x = 0$ is not only stable, but the system's state will also approach zero over time.

Remark 1: A function $V(x)$ that is continuously differentiable and meets the conditions in (2.9) and (2.10) is referred to as a Lyapunov function. Condition based on Lyapunov theory mentioned above is not considered effective because it does not provide a specific method for finding the Lyapunov function. It is the responsibility of the control designer to identify a suitable Lyapunov function that demonstrates the stable behavior around an equilibrium.

2.3.3 System Modeling and Design of the Sliding Mode Surface

We consider the following Linear Time-Invariant (LTI) system without any loss of generality:

$$\dot{x}(t) = Ax(t) + Bu(t). \quad (2.12)$$

Here, $x \in R^n$ represents the dynamic state vector, $u \in R^m$ denotes the system control vector, and $A \in R^{n \times n}$ and $B \in R^{n \times m}$ are constant system matrices.

Assumption 1: It is assumed that $n > m$, the matrix B has full rank m , and the pair (A, B) is fully controllable. In other words, the controllability matrix $[B \ AB \ A^2B \ \dots \ A^{n-1}B]$ has full rank m .

Introduce a sliding variable vector $s(t) \in R^m$ that passes through the origin of the state space.

$$s(t) = Cx(t). \quad (2.13)$$

Here, $C \in R^{m \times n}$ is the sliding mode parameter matrix, and it satisfies $\|CB\| \neq 0$.

System (2.12) is considered to reach the sliding manifold defined by the state vector arrives at and stays on the intersection of the m switching planes.

The equivalent control method is used to describe the system's behavior constrained to the sliding mode surface $s(x) = 0$. On this surface, both $s(x) = 0$ and $\dot{s}(x) = 0$ true. By applying equations (2.12) and (2.13), we obtain:

$$\dot{s}(t) = C\dot{x}(t) = 0, \quad (2.14)$$

$$C(Ax(t) + Bu_{eq}(t)) = 0, \quad (2.15)$$

where $u_{eq}(t)$ is interpreted as the corresponding equivalent control input.

Based on equation (2.14), the equivalent control is derived as follows.

$$u_{eq}(t) = -(CB)^{-1}CAx(t). \quad (2.16)$$

By substituting equation (2.16) into equation (2.12), we obtain the following differential equation

$$\dot{x}(t) = [I - B(CB)^{-1}C]Ax(t). \quad (2.17)$$

Remark 2: The dynamic behavior of the equivalent system does not depend on the control input. Therefore, the matrix C can be determined without requiring prior knowledge of the control input's structure. Typically, the sliding matrix C is designed so that the system response restricted to the sliding surface defined in equation (2.17) exhibits desirable properties, such as asymptotic stability and a specified transient performance. In line with linear control theory, to ensure that the solution of the differential equation in (2.17) is asymptotically stable, the sliding mode parameter vector C must be selected such that all eigenvalues of equation (2.17) possess negative real parts. Furthermore, although the sliding surface in equation (2.13) is linear, it can also be designed in nonlinear forms to achieve finite-time convergence of the system dynamics within the sliding mode.

Remark 3 (Reduction of system dimensionality): Since $s(t) = 0$ holds in the sliding mode, and the matrix C has full rank m , there exist m components of the state vector that can be expressed as functions of the remaining $(n - m)$ components, denoted as $x_2 = s_0(x_1)$. Consequently, the order of the sliding mode dynamics can be reduced by m , resulting in a reduced-order system of the form $\dot{x}_2 = f_1(x_1, t, s_0(x_1))$, with $f_1 \in R^{n-m}$.

2.3.4 Reaching Condition

In essence, ensuring that the switching states attain the sliding manifold constitutes a convergence problem. Consequently, Lyapunov's direct method is commonly employed in sliding mode control (SMC) design as a stability criterion to guarantee that the control switching variable converges to the sliding manifold in the reaching phase. Very frequently,

the following Lyapunov function candidate is adopted in the design of sliding mode controllers.

$$V(t) = \frac{1}{2} s^T(t) s(t). \quad (2.18)$$

To ensure the asymptotic stability of system (7) at the equilibrium point $x(t) = 0$, the following condition for reaching must be fulfilled:

$$\dot{V}(t) = s^T(t) \dot{s}(t) < 0 \text{ for } s(t) \neq 0. \quad (2.19)$$

Remark 4: Condition (2.19) serves as a sufficient criterion for the existence of the sliding mode. It is important to highlight that most sliding mode controllers are developed based on this reachability condition to guarantee that the sliding variable $s(t)$ asymptotically converges to zero.

2.3.5 Reaching Phase Dynamics

The SMC can be formulated using reaching laws to ensure the presence of the sliding mode. Several potential forms of reaching laws are presented in [32]. In general, the reaching law can be expressed in the following generalized form.

$$\dot{s} = -\varepsilon \text{sign}(s) - f(s), \varepsilon > 0 \quad (2.20)$$

where $f(0) = 0$ and $sf(s) > 0$ when $s \neq 0$.

Constant Rate Reaching Law

$$\dot{s} = -\varepsilon \text{sign}(s), \varepsilon > 0 \quad (2.21)$$

This law ensures that the switching variable approaches the switching manifold at a constant rate ε . Its main advantage lies in its simplicity. However, if ε is set too low, the reaching time becomes excessively long, whereas a large ε may lead to pronounced chattering.

Exponential Rate Reaching Law

$$\dot{s} = -\varepsilon \text{sign}(s) - ks, \varepsilon > 0, k > 0. \quad (2.22)$$

Including the proportional term $-ks$ accelerates the convergence of the system states toward the switching manifold, especially when s has a large magnitude.

Power-Rate Reaching Law

$$\dot{s} = -k|s^\alpha| \text{sign}(s), 1 > \alpha > 0, k > 0. \quad (2.23)$$

This reaching phase condition accelerates the convergence when the system states are significantly separated from the sliding mode surface but slows down the convergence rate as the states get closer to the manifold. The three reaching laws discussed above fulfill the reaching condition, thereby guaranteeing the condition under which sliding mode occurs. Notably, a reaching law method serves multiple purposes: it ensures the reaching condition is met, influences the system's dynamic behavior during the reaching phase, and helps control the level of chattering. Therefore, reaching law techniques can be effectively applied to both linear and nonlinear sliding mode control systems, even in the presence of system uncertainties and external disturbances, to enhance the performance during the reaching phase and minimize chattering amplitude.

2.3.6 Equivalent Controller Design

Most VSC approaches generally employ a control input made up of two distinct parts, as described below

$$u(t) = u_{eq}(t) + u_s(t). \quad (2.24)$$

Here, the linear component $u_{eq}(t)$ is defined as in equation (2.11), while the nonlinear part includes the discontinuous term described below.

$$u_s(t) = -\eta(CB)^{-1} \text{sign}(s(t)). \quad (2.25)$$

where $\eta > 0$ is a constant gain used for control.

By substituting equations (2.11) and (2.20) into equation (2.14), we obtain

$$\begin{aligned} \dot{V}(t) &= s^T(t)CAx(t) + s^T(t)CBu(t) \\ &= -\eta|s(t)| < 0. \end{aligned} \quad (2.26)$$

Equation (2.26) confirms that the sliding mode variable will reach the sliding surface within a finite amount of time.

Remark 5: Once the sliding variable vector $s(t)$ is driven to zero, the closed-loop system behavior is governed solely by the desired dynamics described in equation (2.12). As a result,

the system becomes maintains performance despite uncertainties on the sliding mode surface. This inherent robustness to system uncertainties makes SMC a powerful and appealing approach for controlling uncertain systems, inspiring extensive research in the field. However, it is important to note that the system is still influenced by perturbations during the reaching phase—that is, before the sliding surface is attained.

2.3.7 Robustness Property

The robustness property is a key characteristic of SMC systems. In designing an SMC controller, system uncertainties and external disturbances are always considered. By incorporating these factors, the linear time-invariant (LTI) system described in equation (2.12) can be extended and represented in a more general form.

$$\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)u(t) + d(t). \quad (2.27)$$

Here, ΔA and ΔB represent the system uncertainties, while $d(t)$ denotes the external disturbances.

Equation (2.27) can be reformulated as follows:

$$\dot{x}(t) = Ax(t) + Bu(t) + f(t). \quad (2.28)$$

where $f(t) = \Delta Ax(t) + \Delta Bu(t) + d(t)$ represents the total uncertainty.

Building on the concept of equivalent control discussed in Section 2.3.3, the controller can be designed based on the form in equation (2.24), where $u_s(t)$ is defined as in equation (2.25), and the equivalent control is expressed by the following equation.

$$u_{eq}(t) = -(CB)^{-1}C(Ax(t) + f(t)). \quad (2.29)$$

By substituting equation (2.29) into equation (2.28), we obtain the equivalent system equation under sliding mode conditions.

$$\dot{x}(t) = [I - B(CB)^{-1}C]Ax(t) + [I - B(CB)^{-1}C]f(t). \quad (2.30)$$

If $f(t)$ satisfies the matching condition, meaning $f(t) = Bg(t)$, then equation (2.30) reduces to equation (2.17), which is entirely unaffected by system uncertainties and external

disturbances. In other words, the SMC system demonstrates strong robustness against matched uncertainties and disturbances. This invariance characteristic makes SMC a powerful approach for managing uncertain systems and continues to drive significant research interest in the control field. However, since the equivalent control action described in equation (2.29) depends on an unknown external signal, it cannot be implemented directly in practical applications.

2.4 Second Order Sliding Mode Control

In the early 1980s, the control community recognized that one of the major limitations of SMC was the issue of "chattering." To address this problem within the sliding mode framework, A. Levant introduced the concept of Second Order Sliding Mode (SOSM) in [33-34].

The earliest and most straightforward SOSM algorithm is known as the "twisting algorithm" (TA). Let us consider a dynamic system described by the following form:

$$\dot{x} = a(t, x) + b(t, x)u, \quad (2.31)$$

where $x \in R^n$ represents the measurable state vector, and $u \in R$ denotes the control input.

Construct an appropriate sliding surface within the state space.

$$\sigma = \sigma(t, x). \quad (2.32)$$

It is assumed that the system has a relative degree of one, meaning the first derivative of σ can be written as:

$$\dot{\sigma} = h(t, x) + g(t, x)u, \quad h = \dot{\sigma}|_{u=0}, \quad g = \frac{\partial}{\partial u} \dot{\sigma} \neq 0, \quad (2.33)$$

where g and h are certain unknown but smooth functions.

Assume that the conditions defined in terms of input and output

$$0 < K_m \leq g \leq K_M, \quad |h| \leq C \quad (2.34)$$

apply globally for some positive values of k_m , k_M , and $C > 0$.

The modified switching algorithm

$$u = -k \text{sign}(\sigma). \quad (2.35)$$

under the condition

$$0 < K_m \leq g \leq K_M, \quad |h| \leq C \quad (2.36)$$

It can be applied to address the challenge of achieving and maintaining $\sigma = 0$, relying on the known signs of both σ and its derivative $\dot{\sigma}$.

To mitigate chattering, define $v = \dot{u}$ as a new control variable. Differentiating equation (2.28) yields

$$\ddot{\sigma} = h_1(t, x, u) + g(t, x)\dot{u}. \quad (2.37)$$

where

$$h_1 = \dot{h} + \frac{\partial h}{\partial x} \dot{x} + \left(\dot{g} + \frac{\partial g}{\partial x} \dot{x} \right) u.$$

Furthermore, let us define the function

$$\Sigma = \dot{\sigma} + \beta |\sigma|^{\frac{1}{2}} \text{sign}(\sigma). \quad (2.38)$$

Let,

$$v = \dot{u} = \begin{cases} -u & |u| > k \\ -\alpha \text{sign} \Sigma & |u| \leq k \end{cases} \quad (2.39)$$

with a sufficiently large value of α , controller (2.39) guarantees that the finite-time stable Second-Order Sliding Mode $\sigma = 0$ is attained.

Remark 6: The core concept of Second Order Sliding Mode (SOSM) is to influence the second derivative of the sliding variable σ , rather than just its first derivative as in conventional SMC. In SOSM, the control input is taken as the time derivative of the controller, $v = \dot{u}$, rather than the control signal u itself. In essence, the new control strategy involves a discontinuous signal whose integral remains continuous, effectively eliminating chattering.

2.4.1 Higher Order Sliding Mode Control

The initial design of an arbitrary-order sliding mode (SM) controller was proposed in 2001, as noted in [35], allowing for the enforcement of an r –th order sliding mode in finite time while addressing system uncertainties. For a system with output relative degree r , Higher-Order Sliding Mode (HOSM) controllers are designed using a recursive approach. The earliest type of HOSM controller, known as the nested controller [36], follows this recursive structure. Let p represent the least common multiple of the integers 1 through r . Also, let.

$$\begin{aligned} \varphi_{0,r} &= \sigma, N_{1,r} = |\sigma|^{\frac{r-1}{r}} \\ \varphi_{i,r} &= \sigma^{(i)} + \beta_i N_{i,r} \text{sign}(\varphi_{i-1,r}), N_{i,r} = (|\sigma|^{\frac{p}{r}} + \dots |\sigma|^{i-1} |\sigma|^{\frac{p}{r-i+1}})^{\frac{r-i}{p}}. \end{aligned} \quad (2.40)$$

for $i = 1, \dots, r-1$, and considering a sliding mode controller of order r

$$u = -\alpha \text{sign}(\varphi_{i,r}(\sigma, \dot{\sigma}, \dots, \sigma^{(r-1)})). \quad (2.41)$$

when applied to system (2.31), the algorithm ensures that $\sigma, \dot{\sigma} = 0$, and its higher-order derivatives up to $\sigma^{(r-1)}$ are stabilized in finite time, thereby confirming the presence of an r -th order sliding mode

$$\sigma = \dot{\sigma} = \dots = \sigma^{(r-1)} = 0. \quad (2.42)$$

Remark 7: The Higher-Order Sliding Mode (HOSM) method enables finite-time stabilization of a system's output. Nevertheless, certain issues with HOSM controllers remain unresolved, such as: (i) attempting to adapt the controller's gain disrupts the system's homogeneity properties, which are crucial for proving convergence. As a result, the controller's gain cannot be reduced as the system nears the origin. (ii) Additionally, the time required for finite-time convergence increases without bound as the magnitude of the initial conditions grows. (iii) Furthermore, HOSM controllers and differentiators only guarantee asymptotic accuracy, and the constants required to estimate this accuracy still need to be determined [37].

2.4.2 Finite-Time Sliding Mode Control

A newly developed method known as finite-time sliding mode control has recently gained significant attention for its ability to ensure that system dynamics reach the terminal-phase sliding control within a finite time [38]. Compared to the conventional linear sliding mode-based SMC design, Finite-Time Sliding Mode (FTSM) offers the advantage of rapid and finite-time convergence, significantly enhancing high-precision control by increasing the convergence speed near the equilibrium point.

Let us examine the following uncertain nonlinear system of second order:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x) + g(x)u \end{aligned} \quad (2.43)$$

In this context, the system state vector is given by $x = [x_1, x_2]^T$, where $f(x)$ and $g(x) \neq 0$ are continuously differentiable nonlinear functions of x , and u represents the single-valued control input.

To achieve terminal convergence of the system state variables, the first-order terminal sliding variable is introduced as follows:

$$s = x_2 + \beta x_1^{\frac{q}{p}}. \quad (2.44)$$

Here, β is a predefined positive constant, and p and q are positive odd integers that fulfill the following condition:

$$p > q. \quad (2.45)$$

A sufficient condition for the existence of Finite-Time Sliding Mode (FTSM) is given by:

$$\frac{1}{2} \frac{d}{dt} s^2 < -\eta |s|. \quad (2.46)$$

Here, $\eta > 0$ is a constant. According to [38], in the case where $s(0) \neq 0$, the time t_r for the system states to reach the sliding mode $s = 0$ is finite and satisfies the following condition:

$$t_r \leq \frac{|s(0)|}{\eta}. \quad (2.47)$$

To guarantee that the finite-time sliding variable s reaches the finite-time sliding mode surface $s = 0$, we use the following sliding mode controller:

$$u = -g^{-1}(x)(f(x) + \beta \frac{p}{q} x_1^{\frac{p}{q}-1} x_2 + \eta \text{sign}(s)). \quad (2.48)$$

In finite-time sliding mode, the system dynamics are governed by the following nonlinear differential equation:

$$\dot{x}_1 = -\beta x_1^{\frac{p}{q}}. \quad (2.49)$$

It has been demonstrated in [39-42] that $x_1 = 0$ is the terminal attractor of the system (2.49). The finite time t_s required for the system to transition from $x_1(t_r) \neq 0$ to $x_1(t_s + t_r) = 0$ is then given by:

$$t_s = -\beta^{-1} \int_{x_1(t_r)}^0 \frac{dx_1}{x_1^{\frac{p}{q}}} = \frac{p}{\beta(p-q)} |x_1(t_r)|^{1-\frac{q}{p}}. \quad (2.50)$$

According to Expression (2.50), the system states x_1 and x_2 achieve zero values in a finite time under finite-time sliding mode.

Remark 8: The above analysis shows that FTSM provides an outstanding feature by guaranteeing zero-error convergence of the closed-loop system within a finite time. Over the years, this concept has been extensively explored to improve the convergence speed, as seen in Fast Terminal Sliding Mode (FTSM) [43], and to address singularity issues in TSM systems through Non-singular Terminal Sliding Mode (NTSM) [44]. Despite the widespread application of the TSM technique in controlling mechanical, electrical, aircraft, and other

complex systems [45–50], its development remains in the early stages, and much theoretical research is still required in the future.

2.4.3 Integral Sliding Mode Control

Integral sliding modes [51] were introduced to accomplish the following goals.

- The compensation of matched disturbances right from the initial time, meaning that the sliding mode is established from the very beginning. In other words, the reaching phase is avoided.
- Maintaining the original system's dimension, meaning that the system dynamics designed for the ideal, disturbance-free case are preserved.

Assume that a control law, which achieves the control objective, is already designed for an ideal, nominal system.

$$\dot{x} = f(x, t) + B(x)u, \quad x \in R^n, u \in R^m \quad (2.51)$$

Now consider a system affected by disturbances

$$\dot{x} = f(x, t) + B(x)(u + \delta) + \emptyset. \quad (2.52)$$

where δ represents a matched disturbance and $\emptyset$ an unmatched disturbance. A sliding mode control law $u_1(x, t)$ can then be readily incorporated to ensure that the closed loop system

$$\dot{x} = f(x, t) + B(x)(u_0 + u_1 + \delta) + \emptyset. \quad (2.53)$$

is robust against δ .

The process starts by defining the sliding variable.

$$s(x, t) = g(x) - z(t), s(x, t) \in R^m. \quad (2.54)$$

where

$$z(t) = g(x_0) + \int_{t_0}^t g(x)[f(x, \tau) + B(x)u_0(x, \tau)]d\tau, \quad G(x) = \frac{\partial g}{\partial x}(x). \quad (2.56)$$

The product $G(x)B(x)$ forms an invertible function.

Observe that at $t = t_0$, we have $s = 0$, meaning the system begins on the sliding surface (eliminating the reaching phase). Now, let's calculate the time derivative of s :

$$\begin{aligned}\dot{s} &= G(x)[f(x, t) + B(x)(u_0 + u_1 + \delta) + \emptyset - f(x, t) - B(x)u_0] \\ &= G(x)B(x)(u_1 + \delta) + G(x)\emptyset.\end{aligned}\tag{2.57}$$

It is evident that if δ and $\emptyset$ are bounded by known functions, a control input u_1 can be designed to guarantee $\dot{s} = 0$. The corresponding equivalent control is:

$$u_{eq} = -\delta - (G(x)B(x))^{-1}G(x)\emptyset.\tag{2.58}$$

Thus, the system trajectories on the sliding surface are described by:

$$\dot{x} = f(x, t) + B(x)u_0 + [I - B(x)(G(x)B(x))^{-1}G(x)]\emptyset.\tag{2.59}$$

indicating that the system is insensitive to δ .

Remark 9: The influence of $\emptyset$ cannot be completely removed, only reduced. In other words, the projection matrix $G(x)$ should aim to minimize, rather than amplify, the remaining perturbation in (2.59). Techniques like H_∞ optimal control can be employed to further suppress $\emptyset$.

2.4.4 Sliding Mode Control Incorporating Perturbation Estimation

The traditional form of SMC presents several challenges: (i) the designer must know the bounds of the perturbations in advance, which may be difficult or even impossible to determine in practice; (ii) using these bounds to design robust control often leads to overly conservative feedback gains.

Sliding Mode Control with Perturbation Estimation (SMCPE) was proposed in [52–54] to address these issues by leveraging the 'Time-Delayed Control' concept introduced by Youcef-Toumi in 1990. This approach involves online estimation of perturbation effects based on observations of the system dynamics. The approach relies on real-time estimation of perturbation effects by monitoring the system dynamics.

Consider the following dynamic system as an example:

$$\dot{x}(t) = Ax(t) + Bu(t) + f(t). \quad (2.60)$$

where $x(t) \in R^n$ represents the system state vector, $u(t) \in R$ is the control input, and $f(t)$ denotes the lumped perturbation.

The sliding hyperplane is chosen based on Hurwitz polynomials of the system states.

$$s(t) = Cx(t). \quad (2.61)$$

Based on (2.60), the system's actual perturbation at any given time is

$$f(t) = \dot{x}(t) - Ax(t) - Bu(t). \quad (2.62)$$

If all components of the dynamics vary more slowly compared to the loop closure (sampling rate), the right-hand side of (2.62) can be rewritten using $u(t - \delta)$ instead of $u(t)$. This modification provides an up-to-date approximate estimation of the effects of all perturbations $f(t)_{est}$. Thus, (2.62) can be reformulated as:

$$f(t)_{est} = \dot{x}(t) - Ax(t) - Bu(t - \delta). \quad (2.63)$$

where

$$\dot{x}(t) = \frac{x(t) - x(t - \delta)}{\delta}. \quad (2.64)$$

It is demonstrated that the controller structure of

$$u(t) = (CB)^{-1}(-ps - Ksign(s) - CAx(t) - Cf(t)). \quad (2.65)$$

ensures the satisfaction of the reachability condition (2.19) and produces the desired dynamics of s .

$$\dot{s} = -Ps - Ksign(s) + f - f_{est}. \quad (2.66)$$

If $|f - f_{est}|$ stays within a bound of $\eta|f_{est}|$, where $\eta > 0$, the boundary attractivity condition of (2.20) is guaranteed by choosing:

$$K > \eta|f_{est}|_{max} \quad (2.67)$$

2.5 Controller Design and Stability Analysis

The stability of SMC systems has been explored over many years. Unlike continuous-time SMC systems, discrete-time SMC systems exhibit distinct characteristics. In the continuous-time SMC system, the sliding mode existence condition is $s\dot{s} < 0$, whereas in the discrete-time SMC system, this condition no longer holds.

In [55], the discrete SMC problem was initially addressed, and the equivalent of the continuous-time sliding mode existence condition was formulated to establish a discrete-time sliding mode existence condition.

$$(s(k+1) - s(k))x(k) < 0. \quad (2.68)$$

In [56], Milosavljevic introduced the concept of quasi-sliding mode (QSM). QSM refers to motion that meets the following conditions:

- (i) Regardless of the initial state, the trajectory will monotonically approach the switching plane and intersect it within a finite time.
- (ii) After the system's trajectory crosses the switching plane for the first time, it will continue to cross it at each subsequent sampling time, producing a zigzag motion around the switching plane.
- (iii) The magnitude of each subsequent zigzag step remains constant, ensuring that the trajectory stays within a defined band.

It was determined that condition (2.68) is not sufficient for the existence of a QSM. In [57], a new sufficient condition was proposed:

$$|s(k+1)| < |s(k)| \quad (2.69)$$

which was expressed as the following set of inequalities

$$\begin{aligned} (s(k+1) - s(k))\text{sign}(x(k)) &< 0 \\ (s(k+1) + s(k))\text{sign}(x(k)) &> 0 \end{aligned} \quad (2.70)$$

Gao [58] developed a more efficient approach, referred to as the reaching law approach. This method is described as follows:

$$s(k+1) - s(k) = -qTs(k) - \varepsilon T \text{sign}(s(k)), \varepsilon > 0, q > 0, 1 - qT > 0 \quad (2.71)$$

For control law design, Drakunov and Utkin [59] introduced the concept of discrete-time equivalent control, which drives the states onto the sliding surface within one sampling period. To stay on the surface, the corresponding control appears to be nonswitching. Later, a formal definition of sliding mode for discrete-time systems was provided, grounded in the context of semigroups [60]. Soon after, Su et al. [61] proposed a control strategy that ensures the states remain on the switching surface at each sampling instant. Between samples, the states are permitted to deviate from the surface, rather than being precisely on it, even though the equivalent states stay within a boundary layer around the surface.

Chapter 3: GENERAL METHOD OF NONLINEAR DYNAMICAL SYSTEM

3.1 Introduction

In recent decades, sliding mode control has been thoroughly explored and successfully utilized to control complex systems with uncertain behaviors. Typically, when the bounds of system uncertainties and external disturbances are known, a fast-switching sliding mode controller can be developed to guide the closed-loop system onto the sliding surface and keep it there, guaranteeing the desired dynamics with zero-error convergence. However, full knowledge of the unmodeled dynamics is often unattainable in advance, making it inadequate for designing traditional sliding mode controllers. Moreover, selecting excessively large uncertainty bounds can lead to high control signal gains, potentially exceeding the capabilities of the actuators. Switching or chattering of sliding mode control signals as they cross the sliding surfaces is another inherent characteristic of all existing sliding mode control systems. It is widely recognized that this issue has significantly limited the practical applications of sliding mode control, as the high-frequency chattering signals can trigger unwanted high-frequency modes within the closed-loop system.

While the boundary layer method has been commonly employed to reduce chattering, it compromises the zero-error convergence property, as the substituted sigmoid function only ensures the closed-loop system's stability within the defined boundary layer.

3.2 Methodology

Design of Sliding Mode Controller for Second Order differential equation

Consider the following system [20],

$$\ddot{x} = f(x, t) + d(t) + u. \quad (3.1)$$

where f is a nonlinear function, d is time-varying disturbance and u is the control input

We can separate f and d into known parts (with subscript “ m ” for model) and unknown parts (denoted by Δ)

$$f(x, t) = f_m(x, t) + \Delta f(x, t). \quad (3.2)$$

$$d(t) = d_m(t) + \Delta d(t). \quad (3.3)$$

We write the state of the system as

$$\vec{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}. \quad (3.4)$$

Our goal is to find representation for u in terms of $\vec{x}$ such that $\vec{x} \rightarrow x_d(t)$ asymptotically and $\varepsilon = x - x_d$ approaches zero at an exponential rate.

$$\ddot{x} = f_m(x, t) + d_m(t) + \Delta f(x, t) + \Delta d(t) + u. \quad (3.5)$$

Requirement $\varepsilon \rightarrow 0$ as $t \rightarrow \infty$, $\varepsilon = x - x_d$

Now define the sliding surface as

$$s = \dot{\varepsilon} + \lambda \varepsilon. \quad (3.6)$$

If $s = 0$, then $\dot{\varepsilon} = -\lambda \varepsilon \Rightarrow \varepsilon(t) = e^{-\lambda(t-t_0)} \varepsilon(t_0)$ which implies that $\varepsilon(t) \rightarrow 0$ exponentially as $t \rightarrow \infty$ for any positive value of constant λ

Calculating the 1st derivative of s ,

$$\dot{s} = \ddot{\varepsilon} + \lambda \dot{\varepsilon} = \ddot{x} - \ddot{x}_d + \lambda \dot{\varepsilon} \quad (3.7)$$

$$\Rightarrow \dot{s} = f_m(x, t) + d_m(t) + \Delta f(x, t) + \Delta d(t) + u - \ddot{x}_d + \lambda \dot{\varepsilon} \quad (3.8)$$

The u term seems in first derivative of s , we imply to this property as s has relative degree 1.

From equation (3.5)

$$\ddot{x} = f_m + d_m + \Delta f + \Delta d + u \quad (3.9)$$

From equation (3.6), $s = \dot{\varepsilon} + \lambda \varepsilon$

and

From equation (3.7)

$$\dot{s} = f_m + d_m + \Delta f + \Delta d + u - \ddot{x}_d + \lambda \dot{\varepsilon} \quad (3.10)$$

We select u by cancelling out some known terms.

$$u = -f_m - d_m + \ddot{x}_d - \lambda \dot{\varepsilon} + v \quad (3.11)$$

Here the last term v shall deal with the uncertainties. This results in

$$\dot{s} = \Delta f + \Delta d + v \quad (3.12)$$

We need to have at minimum some information about Δf and Δd ,

for example, the upper bounds such that

$$|\Delta f| \leq \alpha(x, t) > 0 \quad (3.13)$$

$$|\Delta d| \leq \beta(t) > 0 \quad (3.14)$$

Next we select a Lyapunov function candidate, $v = \frac{s^2}{2} \Rightarrow \dot{v} = s\dot{s}$ (3.15)

for v such that $s\dot{s} < 0$

Therefore

$$s\dot{s} = s(\Delta f + \Delta d + v) \leq s(\alpha + \beta)\text{sign}(s) + sv \quad (3.16)$$

Suppose, $v = -(\eta + \alpha + \beta)\text{sign}(s)$

$$\rightarrow s\dot{s} \leq -\eta S\text{sign}(s) < 0 \text{ for } \eta > 0 \quad (3.17)$$

Full expression for control law confirming $s \rightarrow 0$ is

$$u = -f_m - d_m + \ddot{x}_d + \lambda \dot{\varepsilon} - (\eta + \alpha + \beta)\text{sign}(s) \quad (3.18)$$

Worst Case Convergence of Time:

$$s\dot{s} = -\eta S\text{sign}(s) \quad (3.19)$$

$$\rightarrow \dot{s} = -\eta \text{sign}(s) \quad (3.20)$$

$$\rightarrow s(t) = s(0) - \eta(t)$$

$$s(t_1) = 0 \text{ when } t_1 = \frac{s(0)}{\eta} \quad (3.21)$$

t_1 is a worst-case convergence time.

Chapter 4: CHAOS MANAGEMENT AND SYNCHRONIZATION IN THE LIU FINANCIAL SYSTEM VIA SLIDING MODE AND PASSIVE CONTROL

4.1. Introduction

The dynamics of economic systems are important to both macro and microeconomics. Economic and financial systems are progressively growing intricately, and financial sectors' commercial growth varies from inadequate to vast. [16]. Numerous factors have contributed to the complexity of the economic growth and expansion process. A major problem in the field of nonlinear science is the study of nonlinear chaotic dynamical systems [62]. The rate of inflation, the value of shares, and capital demand, along with other dynamic elements, are all part of the system [63]. A financial system can exhibit chaotic behavior, even if it has deterministic characteristics. Because of their extreme sensitivity to starting conditions, chaotic systems can take entirely different paths in response to even small mistakes. As a result, financial systems must attain chaos stability. These characteristics are especially prominent in the Liu dynamical financial system, which offers a powerful yet simplified framework for modeling essential financial interactions. Since Lorenz's breakthrough in 1963, scientists have been very interested in studying and developing chaos [64]. There are numerous chaotic systems that have been developed, including the Lu system [65], Chen system [66], and Rossler system [67]. Liu et al. established a system of multidimensional uncontrolled linear equations that only contained two nonlinear terms [68]. Complex dynamical behaviors, such as irregularity and unpredictable long-term future behavior, are characteristics of chaotic systems. Chaos, as a complex and nonlinear dynamic process, is prevalent across various fields. Current approaches to chaos control include a range of strategies intended to control and stabilize chaotic behavior in dynamical systems. Lyapunov control [69] and other nonlinear control strategies are some of the notable methods. Lyapunov control is based on Lyapunov stability theory, which is concerned with establishing and ensuring the stability of nonlinear systems. A Lyapunov function is used in Lyapunov control to examine and regulate the dynamics of a system. Sliding mode control is a straightforward method for managing chaos that has been successfully used in uncertain systems and control nonlinear systems. Russian scientists developed variable structure

control, also referred to as sliding mode control, in the early 1960s and later documented it in Utkin's monograph in the 1970s [70]. Since then, SMC has been extensively studied and effectively used to solve a wide range of real-world problem areas because of its ease of implementation and adaptability in responding to system fluctuations and uncertainty [71]. SMC's sliding motion is its most appealing feature. The system's fluid movement is effectively restricted to a certain region of the entire state space when in the sliding mode. Therefore, adopting an appropriate feedback control method to explore and evaluate chaos management and synchronization is currently my main area of interest. The stability, sliding mode control [72], passive control [73], and synchronization of the Liu financial dynamical system are all explored in this study.

A simplified depiction of reality, an economic model is intended to produce testable hypotheses regarding economic activity. An essential characteristic of an economic model is that, due to the lack of objective metrics for economic results, it is inherently subjective in its creation. To better comprehend this intricate dynamic of these systems, several financial and economic models have been put out in the literature since Strotz et al. [74] discovered chaos in an economic model. Several models are available, including the new hyperchaotic finance model [75], the IS-LM model (Investment Saving-Liquidity Money) [76], the forced Van der Pol model [77], and numerous others. An extremely intriguing model to depict the dynamics of financial systems was put forth by Ma and Chen in 2001[78]. An intense sensitivity to the initial circumstances of the system's variables and parameters was also discovered by the model's analysis, along with some intriguing dynamics. Several financial variables, including the exchange rate, GDP, interest rate, and production, have memory, meaning that any upcoming changes in these variables are impacted by current and historical changes.

Furthermore, in 2020, Liao et al. [79] introduced a new model to enhance comprehension of financial system dynamics. This novel model considers the fact that investment demand influences the price index, in contrast to the approach put out by Ma and Chen [80]. It was discovered that this financial system's complex behavior is the consequence of three components interacting with one another. In financial systems, chaotic behavior is undesirable because it makes economic and financial predictions difficult and, as a result,

increases investment risk. hereafter, its management in the presence of uncertainty pertaining to the system parameters.

In financial systems, chaos control has been extensively researched in the literature. Numerous methods have been developed to manage chaotic financial systems. One of the other well-known control techniques is sliding mode control, whose dynamic performance is dictated by the specified manifold or sliding surface where control is maintained by a switching structure [81]. This technique enforces the system states to remain on the sliding surface, resulting in discontinuous control. Due to its significant impact on lowering complexity and expense, synchronization is now best implemented using a single state controller. Because it only requires one controller, the passive control approach has been becoming more and more popular for synchronizing and managing chaotic systems. The primary goal of passivity theory is to maintain system stability by putting in place a controller that makes the closed-loop system passive upon the characteristics of the system. The passive control approach has been effectively used in recent years to synchronize hyperchaotic Lorenz systems [82], chaotic systems such as Rikitake [83], unified [84], and others. Since passive control does not depend on exact real-time changes, it ensures robust stability even in the face of modeling uncertainties and external disturbances. Numerous studies have examined the sliding mode and passive control methodologies [81-84]. Some unstable financial systems were introduced in the past ten years. The topological structure, stability, equilibrium points, and Lyapunov exponents are examples of the dynamic behaviors of chaotic finance systems including a detailed investigation of Hopf bifurcation analysis [85]. Using linear feedback control and efficient speed feedback control [86], the chaotic financial systems were brought under control, approaches such as adaptive control [87], passive control [88] time-delayed feedback control [89], the choice of gain matrix control [90], and the revision of gain matrix control [91]. It has proven possible to regulate a fractional-order chaotic finance system by employing a sliding mode control technique [92]. Nonlinear feedback controllers [93], adaptive controllers [94], active controllers [95], and a single controller based on Lyapunov stability theory. The chaotic finance systems are synchronized using linear matrix inequality [96].

Despite the extensive studies on chaotic economic systems, including the Lorenz [97], Lu [98], Chen [99], and Rossler systems [100], limited research has focused on the Ĺiu [101] financial dynamical system—a system with complex nonlinear interactions relevant to financial modeling. While various chaos control methods such as Lyapunov control and passive control [88] have been employed in different financial systems, there is still a lack of robust control strategies specifically designed for the Ĺiu system. Additionally, chaos synchronization in financial systems is crucial for stability, yet most existing studies have not explored the combined effects of stability analysis, SMC, and synchronization in the Ĺiu system under various conditions of uncertainty. In this study, we examine the robust regulation of this chaotic behavior and the chaos in the financial system as described by the Ĺiu dynamical system, which is motivated by the discussions above. To address this instability, the study proposes SMC as a robust method for stabilizing chaos, ensuring the system remains resilient against uncertainties. The primary benefit of SMC is its ability to change the control law, which forces the system's states from their initial states onto a predetermined sliding mode surface [102]. The system on the sliding mode surface has desirable characteristics, including quick response, low sensitivity to outside noises, robustness to system uncertainties, easy realization, and so forth. Around this sliding surface, SMC produces a discontinuous control law and limited switching [20]. Consequently, this sliding surface is bound to determine the direction of the state variables' motion and the dynamics of the system. However, the existence of such a control law leads to an undesirable issue with sliding control, namely the "chattering phenomenon," which should be reduced or eliminated in some way. However, the existence of such a control law leads to an undesirable issue with sliding control, namely the "chattering phenomenon," which should be reduced or eliminated in some way. The literature has proposed a number of methods, most of which are predicated on the continuous approximation of the discontinuous sign function. A sliding mode controller's design consists of:

- 1) Use a "sliding surface" that is robust during the reaching phase and depicts the "desired stable dynamics."
- 2) The "control law" that ensures "the sliding condition" and "the reaching condition" is described.

Additionally, the research explores synchronization between two Ĺiu financial systems, which is crucial for maintaining coherence in financial behavior and stability. A comparative study is conducted to evaluate the effectiveness of Lyapunov control, passive control, and SMC, identifying the most suitable approach for managing chaos in financial dynamics. Furthermore, the study validates its theoretical findings through extensive numerical simulations, including time response analysis, phase portraits, and synchronization error dynamics, demonstrating the practical applicability of the proposed control methods. By addressing these key aspects, this research advances nonlinear financial modeling and offers an effective approach to managing chaos in economic systems, ultimately contributing to improved financial stability.

This chapters remaining sections are organized as follows. In Section 4.2, the Ĺiu financial dynamical system is introduced, and its governing equations are discussed. The system's parameters and their economic interpretations are also provided. Additionally, the chaotic behavior of the Ĺiu system under specific parameter conditions is analyzed. Section 4.2.1 focuses on the synchronization of the Ĺiu dynamical system using SMC. The design of the SMC strategy is presented, followed by a stability analysis based on Lyapunov theory. The effectiveness of the proposed control scheme in achieving synchronization is demonstrated. In Section 4.2.2, the organization of the Ĺiu financial system via Passive Control (PC) is explored. The principles of passive control are introduced, and a control law is designed to stabilize the system. Theoretical justifications for stability are provided, along with a comparison between SMC and PC strategies. Section 4.3 presents numerical simulations to validate the proposed control methods. The time response, phase portraits, and error dynamics are illustrated to confirm the effectiveness of the applied controllers. A comparative analysis between different control strategies is also included. Finally, concludes the chapter with a summary of key findings, highlighting the contributions of the study. Future research directions, including adaptive control techniques and further applications in financial modeling, are also discussed.

4.2. Ĺiu Financial Dynamical System

To foster economic growth and commercial demand through investment, a financial system's organizational units and exchanges usually connect in a complex way. Three state

variables' temporal variation has been identified by the economic model considered in this study: Let x represent the interest rate or inflation rate, y represents investment demand or capital, and z represent the price index or market price. The price exponent determines the variance in price distribution. The quantity that the lender pays the borrower to utilize assets, represented as an amount of capital, is known as the rate of return. The desired or expected to capital and inventory of an organization is known as expenditure demand, and it is negatively correlated with financing rates and investment costs. The Liu financial dynamical system is described by a set of three first-order differential equations as follows [101]:

$$\begin{aligned}\dot{x} &= a(y - x) \\ \dot{y} &= bx - dxz \\ \dot{z} &= -cz + fx^2\end{aligned}\tag{4.1}$$

Where a, b, c, d , and f are all beneficial constant parameters that, respectively, describe the quantity of savings, the cost per savings, the elasticity of economic demand, the influence of the cross market, and the growth factor of investments [103]. A fiscal system's savings level shows that the enterprise unit's overall economic expansion has accelerated. The quantity of distribution from objective investments that is obtained without initial expense is defined as the per-investment cost. The relationship between the cost of a good change and alterations in the quantity demanded is known as advertising demand elasticity. Cross-market influence means the impact of interactions across different markets or sectors, such as how the behavior in one market (supply or demand) influences another. The growth factor of investments means a growth or amplification factor in the system, representing how investments fuel further expansion, potentially due to compounding effects or returns.

In equation (4.1),

Interest Rate Dynamics:

$$\dot{x} = a(y - x)$$

- The interest rate adjusts depending on the gap between investment demand and the current rate.
- If investment demand y is higher than interest rate x , the interest rate rises.

Investment Demand Dynamics:

$$\dot{y} = bx - dxz$$

- Investment demand grows with interest income (bx), but is dampened by the cost of borrowing and price inflation (represented by the nonlinear dxz term).

Price Index Dynamics:

$$\dot{z} = -cz + fx^2$$

- The price index (e.g. inflation) decays naturally over time ($-cz$) but increases nonlinearly with the square of the interest rate.

When $a = 10, b = 40, c = 2.5, f = 4$, and $d = 1$ the dynamic financial system behaves chaotically [100]. Fig.4.1 displays the chaotic financial system's timing response with the initial circumstances $x(0) = 10, y(0) = 20$, and $z(0) = 90$. Fig. 4.2 displays the 2D phase portraiture, Fig. 4.3 displays the 3D phase plane, and Figure 4.4 displays the error states' time response prior to synchronization.

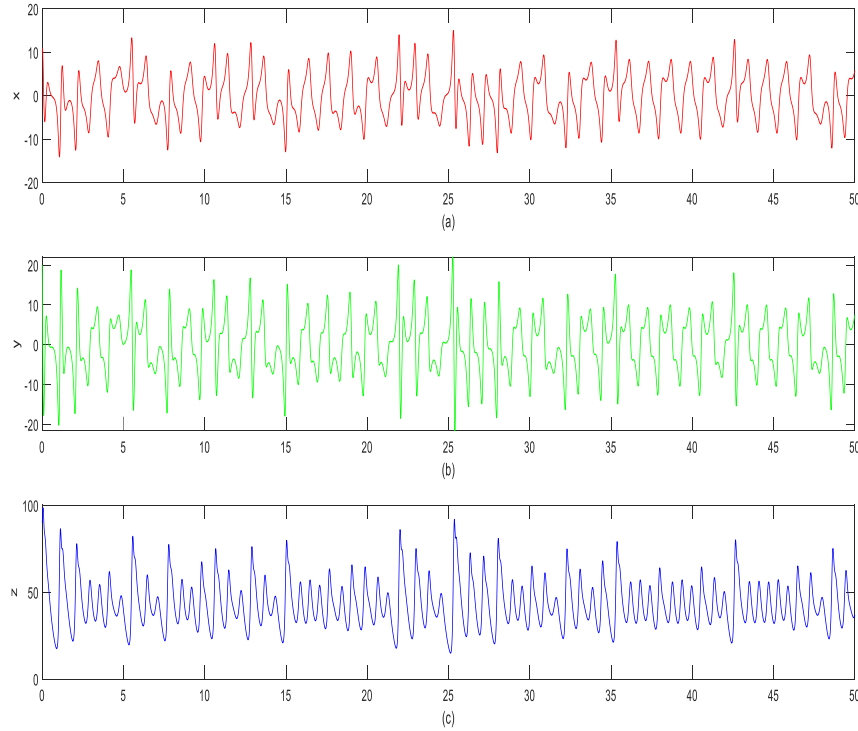


Fig. 4.1: Chaos finance system time signals x , y , and z prior to synchronization for (a) x signals, (b) y signals, and (c) z signals

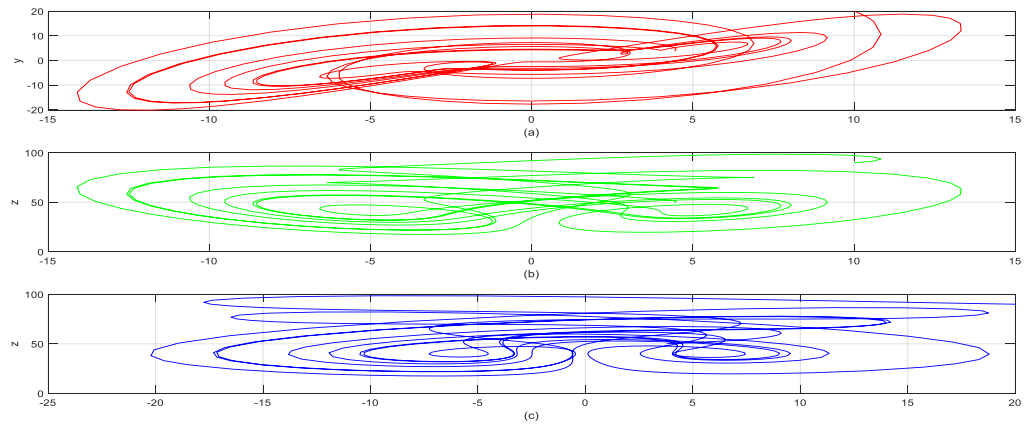


Fig. 4.2: Phase depicts of a chaotic financial system in three different planes: (a) x-y, (b) y-z, and (c) z-x

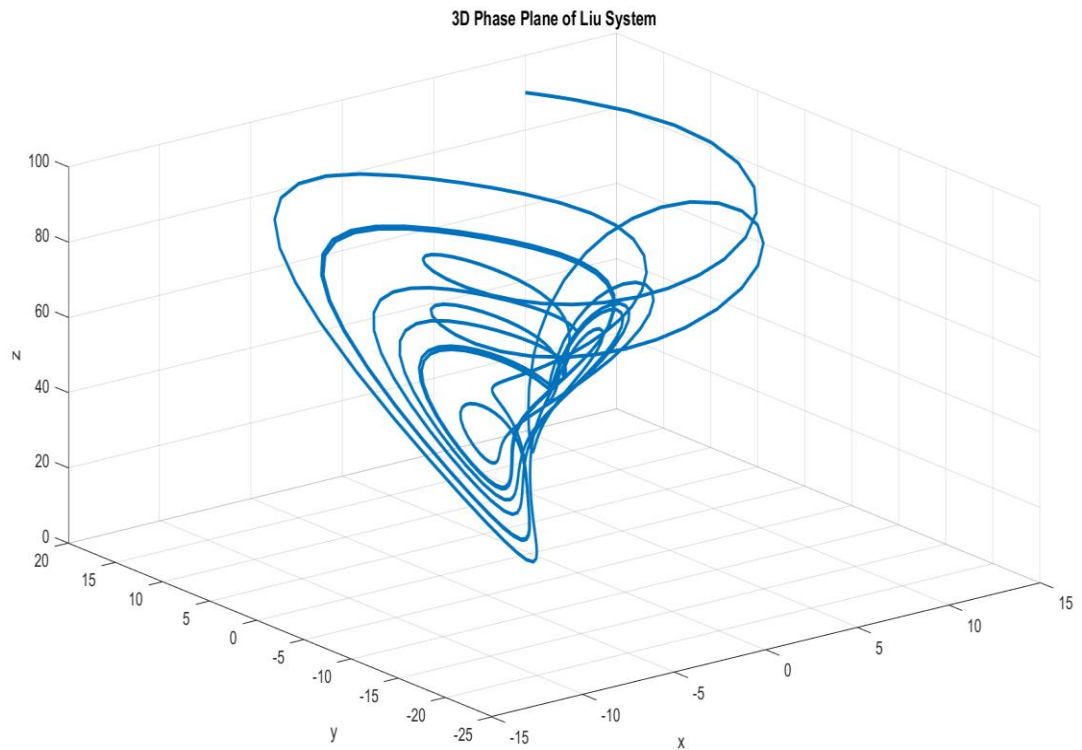


Fig. 4.3: The chaotic finance system's three-dimensional phase plane

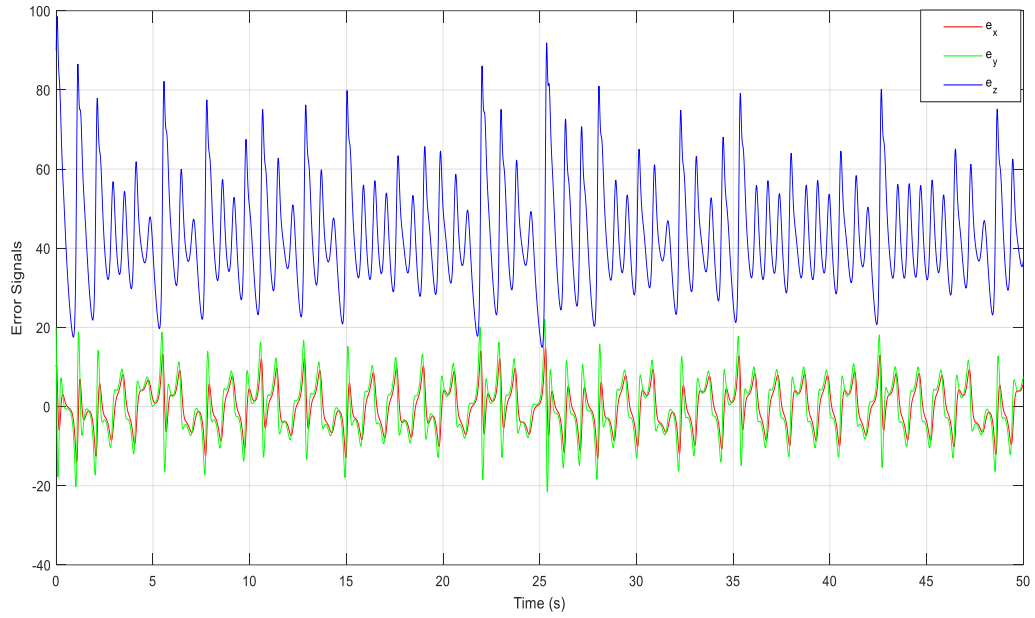


Fig. 4.4: Interval retort of the error states before synchronization.

4.2.1. The Synchronization of Liu Dynamical System Applying SMC

The boundaries a , b , d , c and f are taken within a limit; make certain that the system (4.1) will demonstrate changeable behavior. It is expected that two chaotic financial systems, with the drive system controlling the response system, are used to observe the synchronization. Subtitles 1 and 2 stand for the drive system and response system, respectively. The drive system is provided by,

$$\begin{aligned}\dot{x}_1 &= a(y_1 - x_1) \\ \dot{y}_1 &= bx_1 - dx_1z_1 \\ \dot{z}_1 &= -cz_1 + fx_1^2\end{aligned}\tag{4.2}$$

and the response system is described as

$$\begin{aligned}
\dot{x}_2 &= a(y_2 - x_2) + u_1(t) \\
\dot{y}_2 &= bx_2 - dx_2z_2 + u_2(t) \\
\dot{z}_2 &= -cz_2 + fx_2^2 + u_3(t)
\end{aligned} \tag{4.3}$$

where $u_1(t)$, $u_2(t)$, $u_3(t)$ in Eq. (4.3) are to be formed. To determine the control capabilities for time, the drive system is eliminated from the reaction time system. The state errors e_1 , e_2 , and e_3 are to be controlled and the controlling system (4.2) are expressed as

$$\begin{aligned}
e_1 &= x_2 - x_1 \\
e_2 &= y_2 - y_1 \\
e_3 &= z_2 - z_1
\end{aligned} \tag{4.4}$$

Thus, the patterns of errors become

$$\begin{aligned}
\dot{e}_1 &= ae_2 - ae_1 + u_1(t) \\
\dot{e}_2 &= be_1 - d(x_2z_2 - x_1z_1) + u_2(t) \\
\dot{e}_3 &= -ce_3 + f(x_2^2 - x_1^2) + u_3(t)
\end{aligned} \tag{4.5}$$

In matrix notation, we can normalize the error behavior (4.5) to

$$\dot{e} = Ae + \eta(x, y) + u. \tag{4.6}$$

where,

$$\begin{aligned}
A &= \begin{bmatrix} -a & a & 0 \\ b & 0 & 0 \\ 0 & 0 & -c \end{bmatrix}, \quad \eta(x, y) = \begin{bmatrix} 0 \\ -d(x_2z_2 - x_1z_1) \\ f(x_2^2 - x_1^2) \end{bmatrix} \\
u &= \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}
\end{aligned} \tag{4.7}$$

The meaning of the control signal u established on the SMC methodology is as follows [104]:

$$u(t) = -\eta(x, y) + Bv(t). \quad (4.8)$$

where B is a matrix and v is a control signal. To make (A, B) controllable, B was selected.

Consequently, B is interpreted as

$$B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad (4.9)$$

Substituting (4.8) into (4.7) the error behavior simplifies to,

$$\dot{e} = Ae + Bv. \quad (4.10)$$

It has a single input, v , and is a proportional time-dependent control scheme.

The initial global chaotic synchronizing question can be converted into an analogous problem of stabilizing the system's zero-solution $e = 0$ in equation (4.10), provided that an appropriate SMC is selected. The switchable mode control defines the variable.

$$s(e) = Ce = c_1e_1 + c_2e_2 + \dots + c_ne_n. \quad (4.11)$$

as, $C = [c_1 \ c_2 \ \dots \ c_n]$ is a coefficient trajectory needs to be formed.

Now reduce the system (4.10) motion in the SMC to the sliding manifold denoted by, $s = \{x \in R^n, |s(e)| = 0\}$

which needs to remain unchanged under the influence of the error dynamics described by equation (4.10). When the sliding manifolds, the system (4.10) fulfills the following requirements:

$$s(e) = 0. \quad (4.12)$$

It becomes a series of defining equations and,

$$\dot{s}(e) = 0. \quad (4.13)$$

In the chaotic case, the parameter values, $a = 10$, $b = 40$, $c = 2.5$, $f = 4$ and $d = 1$

The variable for the sliding mode is chosen as

$$S = Ce = [2 \ 0 \ -1]e = 2e_1 - e_3.$$

which ensures that the sliding mode state equation is asymptotically stable.

Select the sliding mode gain as follows, $K = 5$ and $q = 0.1$.

A high value of K can lead to chattering, while a suitable rate of q is selected to both accelerate the time it takes to accomplish the sliding manifold and minimize system chattering. Using (4.10) and (4.11) the equation (4.13) can be changed as

$$\dot{s}(e) = C(Ae + Bv). \quad (4.14)$$

According to the sliding mode control theory's property [100],

$$v(t) = -(CB)^{-1}[C(KI + A)e + q\text{sign}(s)]. \quad (4.15)$$

where C is chosen such that $CB \neq 0$

Now, the $v(t)$ control signal becomes,

$$v(t) = -10e_1 + 20e_2 - 2.5e_3 + 0.1\text{sign}(s). \quad (4.16)$$

As a result, the necessary sliding mode controller is acquired as [104]

$$u = -\eta(x, y) + Bv. \quad (4.17)$$

Then, the required sliding mode control signal is obtained as Eq. (4.8) where $\eta(e)$ and B are described as in Eq. (4.7) and (4.9), respectively:

$$\begin{aligned} u_1(t) &= 0 \\ u_2(t) &= d(x_2 z_2 - x_1 z_1) + v(t) \\ u_3(t) &= -f(x_2^2 - x_1^2) + v(t) \end{aligned} \quad (4.18)$$

Equations (4.16) and (4.17) complete the synchronization of Liu system (4.2.1) using the SMC approach. Therefore, two equal Liu systems work together using SMC.

Theorem 4.2.2. For all initial conditions $x(0), y \in (0)R^n$, the SMC law guarantees that the drive system (4.2) and the response system (4.3) are globally and asymptotically synchronized.

$$u(t) = -\eta(x, y) + Bv(t) \quad (4.19)$$

where B is a column vector such that (A, B) is controllable and $v(t)$ is defined by (4.15). Furthermore, k and q , the sliding mode gains, are positive.

Proof. Firstly, observe that the closed-loop dynamics can be achieved as by putting in (4.19) and (4.17) into the error dynamics (4.7).

$$\dot{e} = Ae - B(CB)^{-1}[C(KI + A)e + qsgn(s)]. \quad (4.20)$$

To demonstrate that the error dynamics (4.20) is globally stable over time, let's examine the proposed Lyapunov function that the equation depicts.

$$v(e) = \frac{1}{2} s^2(e). \quad (4.21)$$

which is a positive definite function on R^n .

Differentiating v along the trajectories of (4.20) or the equivalent dynamics (4.14), we obtain

$$\dot{v}(e) = s(e)\dot{s}(e) = -ks^2 - qsgn(s) \quad (4.22)$$

which is a negative definite function on R^n .

The error dynamics (4.20) is therefore immediately globally asymptotically stable for all initial conditions $e(0) \in R^n$ according to Lyapunov equilibrium principle.

This concludes the documentation.

4.2.2 The Organization of Liu Systems via Passive Control (PC)

At this point, the system that drives will continue to be [105]

$$\begin{aligned} \dot{x}_1 &= a(y_1 - x_1) \\ \dot{y}_1 &= bx_1 - dx_1z_1 \\ \dot{z}_1 &= -cz_1 + hx_1^2 \end{aligned} \quad (4.23)$$

and the system for reporting is provided by,

$$\begin{aligned}
\dot{x}_2 &= a(y_2 - x_2) + u(t) \\
\dot{y}_2 &= bx_2 - dx_2z_2 \\
\dot{z}_2 &= -cz_2 + hx_2^2
\end{aligned} \tag{4.24}$$

where the passive control function to be found is $u(t)$ in equation (4.24). The timing error is obtained by subtracting the drive system from the response system, similar to the SMC. Then,

$$\begin{aligned}
\dot{e}_1 &= ae_2 - ae_1 + u(t) \\
\dot{e}_2 &= be_1 - dx_2z_2 + dx_1z_1 \\
\dot{e}_3 &= -ce_3 + h(x_2^2 - x_1^2)
\end{aligned} \tag{4.25}$$

where system (4.25) is referred to as the error system and e_1, e_2 and e_3 are the actual errors. System (4.25) has a word that may be expressed as

$$x_2^2 - x_1^2 = (x_2 + x_1)e_1 \tag{4.26}$$

So, error system (4.25) can be rewritten in the following form:

$$\begin{aligned}
\dot{e}_1 &= ae_2 - ae_1 + u(t) \\
\dot{e}_2 &= be_1 - dx_2z_2 + dx_1z_1 \\
\dot{e}_3 &= -ce_3 + he_1(x_2 + x_1)
\end{aligned} \tag{4.27}$$

To stabilize the error system (4.27) at zero equilibrium, the passive controllers $u(t)$ must be found.

By assuming that the state variable e_1 is the output of the system and supposing $Y = e_1, z_1 = e_2, z_2 = e_3, z = [z_1, z_2]^T$, the system (4.27) can be denoted by normal form:

$$\begin{aligned}
\dot{Z}_1 &= bY - dx_2z_2 + dx_1z_1 \\
\dot{Z}_2 &= -ce_3 + hY(x_2 + x_1) \\
\dot{Y} &= az_1 - aY + u(t)
\end{aligned} \tag{4.28}$$

The modified iteration of the passive control theory is as follows:

$$\dot{Z} = f_0(Z) + p(Z, Y)Y,$$

$$\dot{Y} = b(Z, Y) + a(Z, Y)u,$$

and according to system (4.28):

$$\begin{aligned} f_0(Z) &= \begin{bmatrix} -dx_2z_2 + dx_1z_1 \\ -cz_2 \end{bmatrix}, \\ p(Z, Y) &= \begin{bmatrix} b \\ h(x_2 + x_1) \end{bmatrix}, \\ b(Z, Y) &= aZ_1 - aY \\ a(Z, Y) &= 1 \end{aligned} \quad (4.29)$$

where $W(Z) = \frac{1}{2}(Z_1^2 + Z_2^2)$ is a Lyapunov function of $f_0(Z)$ with $W(0) = 0$. Then

$$\begin{aligned} \frac{d}{dt}V(Z, Y) &= \frac{\partial W(Z)}{\partial Z} \dot{Z} + Y \dot{Y} \\ &= \frac{\partial W(Z)}{\partial Z} f_0(Z) + \frac{\partial W(Z)}{\partial Z} p(Z, Y)Y + Yb(Z, Y)u \end{aligned} \quad (4.30)$$

According to Eq. (4.29), by taking the derivative of $W(Z)$

$$\begin{aligned} \dot{W}(Z) &= \frac{d}{dt}W(z) = \frac{\partial W(Z)}{\partial Z} f_0(Z) = [Z_1 \ Z_2] \begin{bmatrix} -dx_2z_2 + dx_1z_1 \\ -cz_2 \end{bmatrix} \\ &= \begin{bmatrix} -dx_2z_1z_2 + dx_1z_1^2 \\ -cz_2^2 \end{bmatrix} \end{aligned} \quad (4.31)$$

Since $W(Z) \geq 0$ and $\dot{W}(Z) \leq 0$, it can be decided that $W(Z)$ is the Lyapunov function of $f_0(Z)$ and that $f_0(Z)$ is globally asymptotically stable. The controlled system (4.27) is equivalent to a passive system and can be asymptotically globally stabilized at its zero equilibrium by the following state feedback controller:

$$\begin{aligned} u(t) &= a(Z, Y)^{-1} \left[-b^T(Z, Y) - \frac{\partial W(Z)}{\partial Z} p(Z, Y) - \alpha Y + v \right] \\ &= 1^{-1} [-(az_1 - aY) - [z_1 \ z_2] \begin{bmatrix} b \\ h(x_1 + x_2) \end{bmatrix} - \alpha Y + v] \\ &= aY - (a + b)z_1 - h(x_1 + x_2)z_2 - \alpha Y + v \end{aligned} \quad (4.32)$$

where v is an external input signal and α is a positive constant. $Z_1=e_2$, $Z_2=e_3$, and $Y=e_1$ conversions are noted, and the passive control function turns into

$$u(t) = ae_1 - (a + b)e_2 - h(x_1 + x_2)e_3 - \alpha e_1 + v \quad (4.33)$$

Equation (4.33) completes the synchronization of the chaotic finance system (4.24) using the passive control method. As a result, two identical chaotic finance systems are synchronized through passive control.

4.2.3: The Theory of Passive Control

Consider the following differential equation:

$$\begin{cases} \dot{x} = f(x) + g(x)u, \\ y = h(x), \end{cases} \quad (4.34)$$

where $x \in X \subset R^n$ is the state variable. $f(x)$ and $g(x)$ are the smooth vector fields. $u(t) \in U$ is the input. $f(0) = 0$ and $h(x)$ is a smooth mapping.

Definition 1: The system (4.34) is a minimum phase system if $L_g h(0)$ is nonsingular and $x = 0$ is one of asymptotically stabilized equilibrium points of $f(x)$.

Definition 2: The system (4.34) is passive if the following two conditions are satisfied:

1. $f(x)$ and $g(x)$ exist and are smooth vector fields, $h(x)$ is a smooth mapping
2. For any $t \geq 0$, there is a real value β satisfying the inequality

$$\int_0^t u^T(\tau) d\tau \geq \beta, \quad (4.35)$$

or there are a real value β and $\rho > 0$ satisfying the inequality

$$\int_0^T u^T(\tau) y(\tau) d\tau + \beta \geq \int_0^T \rho y^T(\tau) y(\tau) d\tau. \quad (4.36)$$

Let $z = \vartheta(x)$, and the system (4.34) will be changed into the following generalized form:

$$\begin{cases} \dot{z} = f_0(z) + p(z, y)y, \\ \dot{y} = b(z, y) + a(z, y)u, \end{cases} \quad (4.37)$$

where $a(z, y)$ is nonsingular for any (z, y) .

Theorem 4.2.5: If the system (4.34) is a minimum phase system and has a relative degree $[1,1,\dots]$ at $x = 0$, the system (4.37) will be equivalent to a passive system and asymptotically stabilized at an equilibrium point through the local feedback control as follows:

$$u = a(z, y)^{-1} \left[-b^T(z, y) - \frac{\partial}{\partial z} W(z) p(z, y) - \alpha y + v \right], \quad (4.38)$$

where $W(z)$ is the Lyapunov function of $f_0(z)$, α is a positive real value and v is an external signal that is connected with the reference input.

Proof: Suppose that

$$V(z, y) = W(z) + \frac{1}{2} y^2, \quad (4.39)$$

$$\begin{aligned} \frac{d}{dt} V(z, y) &= \frac{\partial W(z)}{\partial z} \dot{z} + y \dot{y} \\ &= \frac{\partial W(z)}{\partial z} f_0(z) + \frac{\partial W(z)}{\partial z} p(z, y) y + y b(z, y) + y a(z, y) u. \end{aligned} \quad (4.40)$$

Because the system (4.37) is a minimum phase system, the inequality is obtained to be

$$\frac{\partial W(z)}{\partial z} f_0(z) \leq 0, \quad (4.41)$$

So

$$\frac{d}{dt} V(z, y) \leq \frac{\partial W(z)}{\partial z} p(z, y) y + y b(z, y) + y a(z, y) u \quad (4.42)$$

Substituting Eq. (4.38) into the inequality (4.42) we have

$$\frac{d}{dt} V(z, y) \leq -\alpha y^2 + v y \quad (4.43)$$

Performing integration on both sides of the inequality (4.43) yields

$$V(z, y) - V(z_0, y_0) \leq -\int_0^t \alpha y(\tau)^2 d\tau + \int_0^t \alpha y(\tau)^2 d\tau + \int_0^t v(\tau) y(\tau) d\tau \quad (4.44)$$

For $V(z, y) \geq 0$, let $V(z_0, y_0) = \mu$, the above inequality can be written as

$$\int_0^t v(\tau) y(\tau) d\tau + \mu \geq \int_0^t \alpha y(\tau)^2 d\tau + V(z, y) \geq \int_0^t \alpha y(\tau)^2 d\tau \quad (4.45)$$

According to definition 2, the system (4.37) is a passive system. Because $W(z)$ is radially unbounded, it follows from Eq. (4.39) that $V(z, y)$ is also radially unbounded, so that a closed-loop system is in a bounded state stable for $[z^T, y]^T$. This means that we can use the local feedback control described in the form (4.38) to regulate the system (4.37) to an equilibrium point.

4.3. Numerical Simulations

In this portion, Liu's financial dynamical systems are shown to be brought into line using numerical simulations completed with MATLAB. The system is simulated using the fourth-order Runge–Kutta method with a fixed step size of 0.001. The bound values of nonlinear financial systems are required as $a = 10, b = 40, c = 2.5, f = 4$ and $d = 1[100]$. The initial values are selected as $x_1(0)=10, y_1(0)=20, z_1(0)=90, x_2(0)=8, y_2(0)=18, z_2(0)=85$. To minimize chattering, the sliding mode control constant q is set to 0.1. Figs. 4.5 and 4.6 illustrate how the SMC for synchronizing finance dynamical systems activates the states to organize responses at $t = 10$ and $t = 5$, respectively. Figs. 4.7 and 4.8 demonstrate that the states at $t = 10$ and $t = 5$, response times for synchronizing financial dynamical systems with passive controllers are initiated. The error signals of synchronization in sliding mode control are explained in Figs. 4.9 and 4.10. The error signals of synchronization in passive control are shown in Figs. 4.11–4.12. As shown in Figs. 4.5 and 4.7, the outputs indicate that both the SMC and the PC successfully synchronized the chaotic financial systems within an appropriate timeframe. In the comparative results of the SMC and PC, both stimulated at $t = 10$, synchronization is achieved by the SMC $t \leq 1$, while the PC reaches synchronization at $t \leq 13$. The error signals are shown that Figs. 4.7 and 4.11 come together asymptotically to zero. Comparative findings for synchronizing chaotic financial systems are presented in these figures. With controllers activated at $t = 10$, synchronization is achieved by SMC at $t \geq 1$, whereas PC reaches synchronization at $t \geq 13$. Synchronization is initially observed with the SMC once they are started at $t = 5$. While the PC method only needs one controller, the SMC approach uses two controllers to accomplish synchronization. The passive control method achieves synchronization by adjusting the interest rate, either by adding or subtracting a value, based on factors such as the savings amount, current interest rates, and investment demands. It is easier to implement since it does not require adjustments to the price exponent

or investment demand. On the contrary, the sliding mode control method modifies the rate of interest and price index to achieve synchronization. Using the saving amount, per-investment cost, interest rates, investment demands, and price index, it determines the total amount of changes. The elasticity associated with financial demands for synchronization is not necessary for both approaches. Whereas the SMC approach seems to have certain advantages in terms of synchronization speed, it is more difficult to implement than passive control.

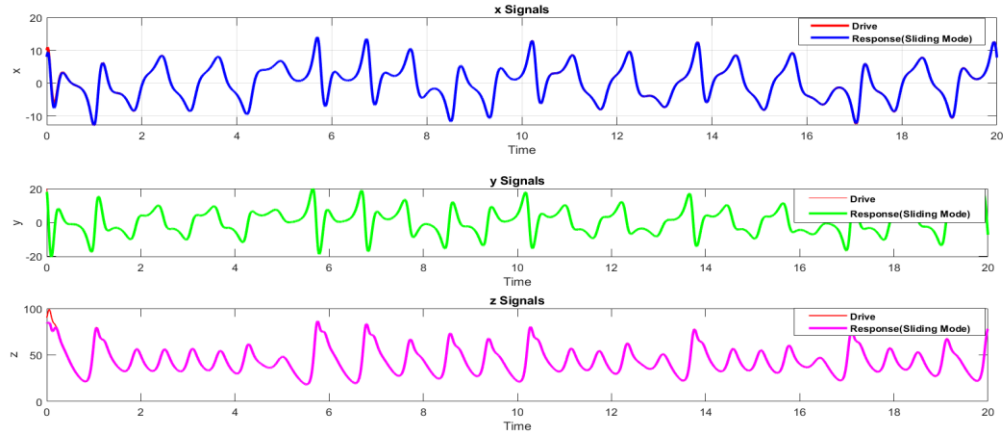


Fig.4.5: The interval retort of states for synchronization when the sliding mode controller is started $t=10$ (a) x signs, (b) y signs, (c) z signs

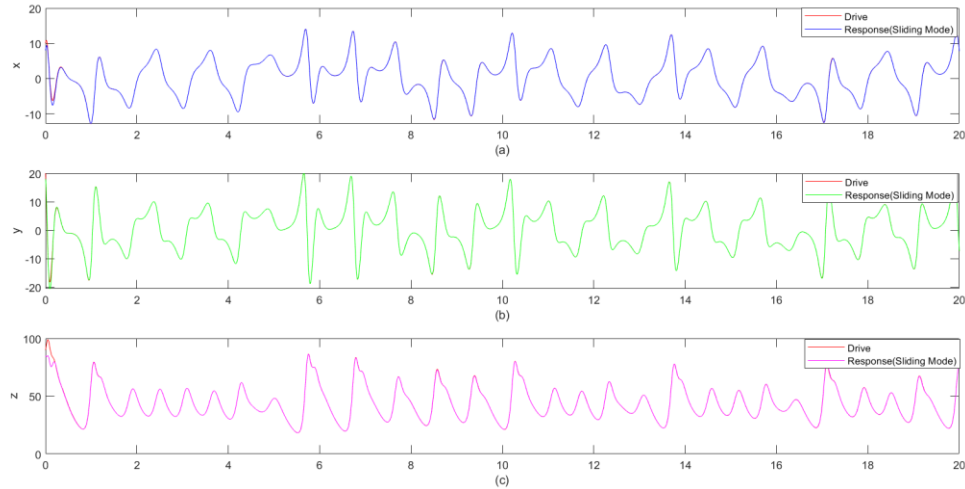


Fig.4.6: The interval retort of states for synchronization when the sliding mode controller is started $t=5$ (a) x signs, (b) y signs, (c) z signs.

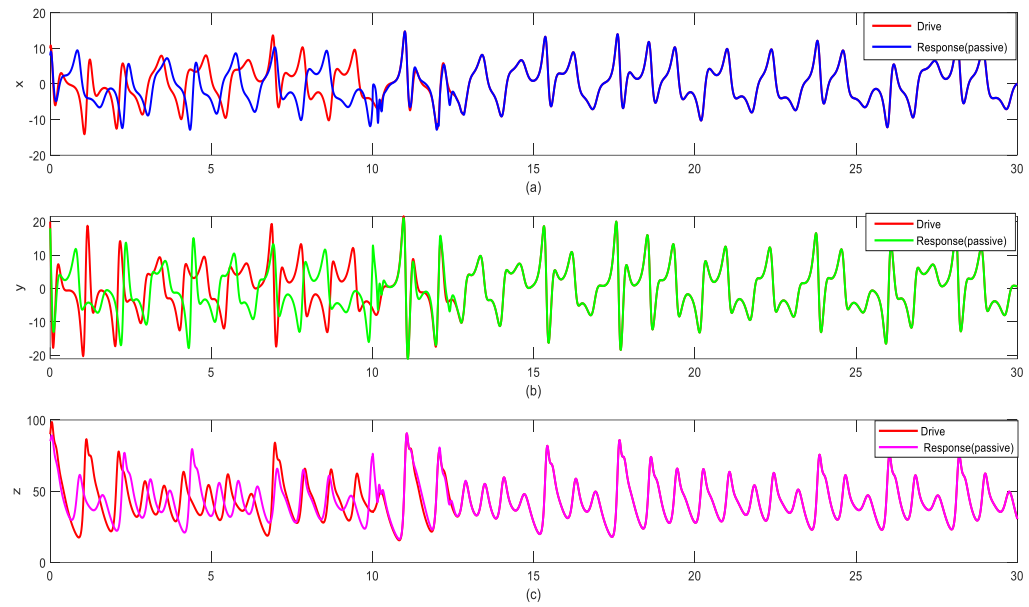


Fig.4.7: The interval retreat of states for synchronization when the passive controller is started $t=10$ (a) x signs, (b) y signs, (c) z signs.

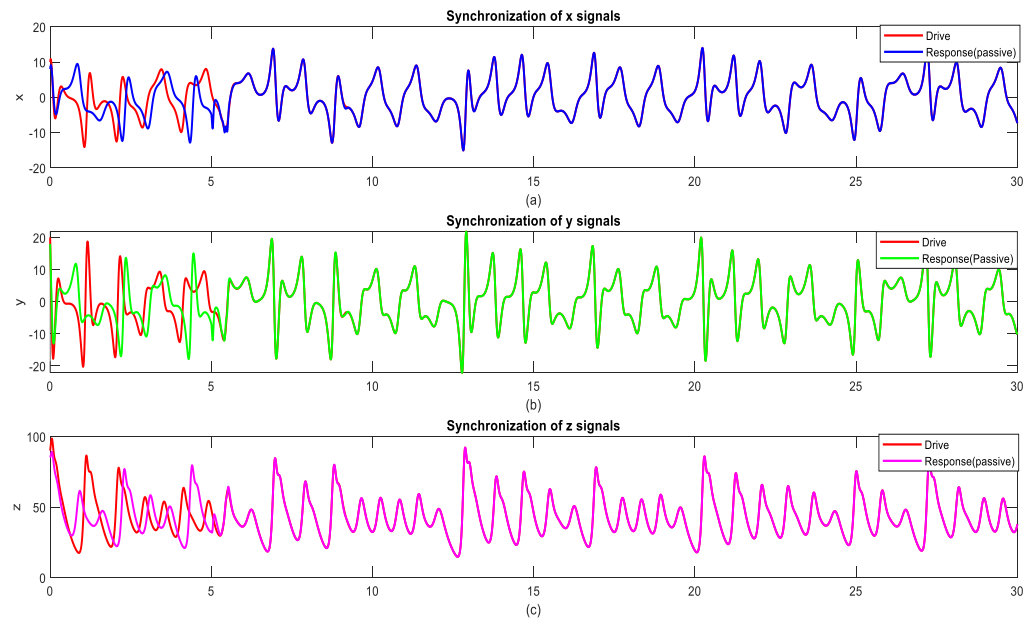


Fig.4.8: The interval retreat of the states for synchronization when the passive controller is started $t=5$ (a) x signs, (b) y signs, (c) z signs

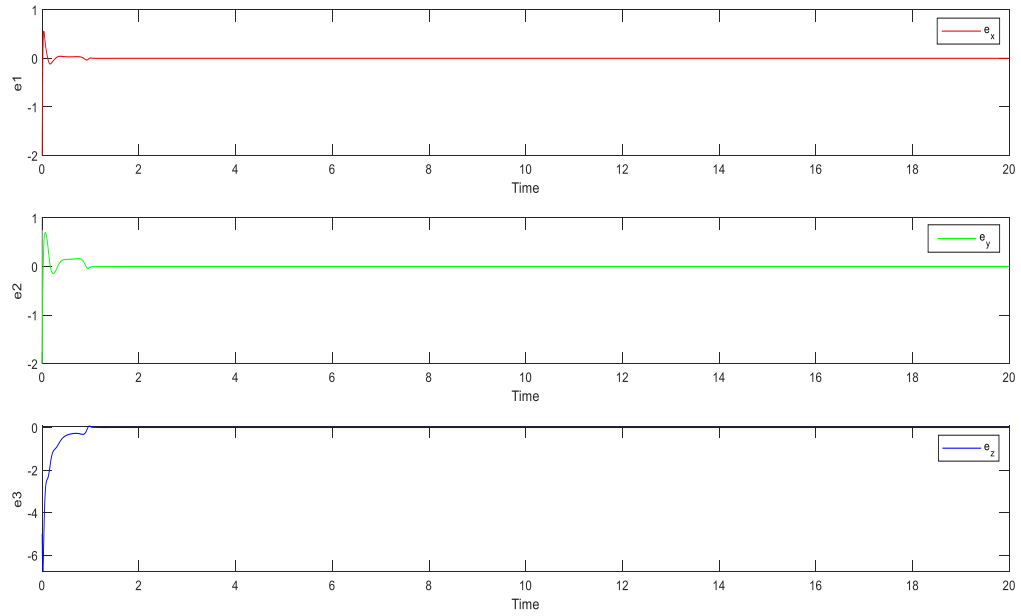


Fig.4.9: The retort of error signs for synchronization when the sliding mode controller is activated $t=10$ (a) e_1 signs, (b) e_2 signs, (c) e_3 signs

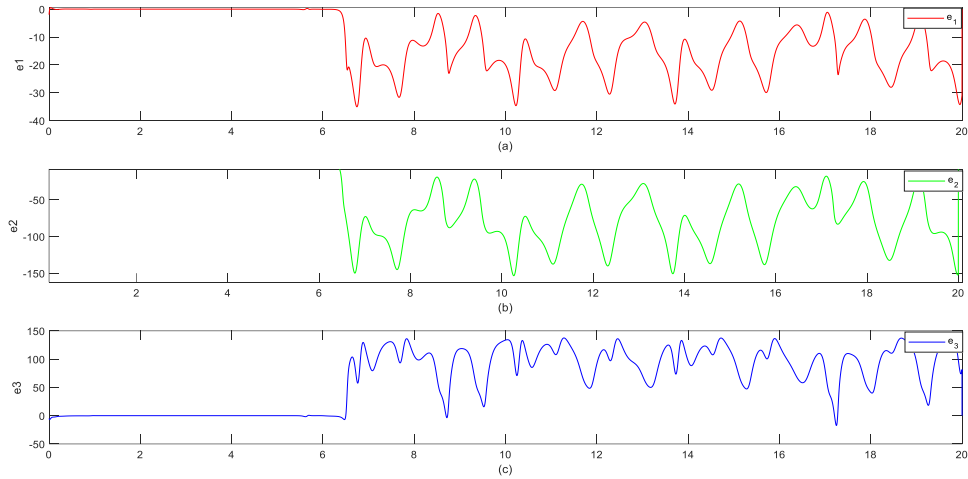


Fig.4.10: The retort of error signs for synchronization when the sliding mode controller is activated $t=5$ (a) e_1 signs, (b) e_2 signs, (c) e_3 signs.

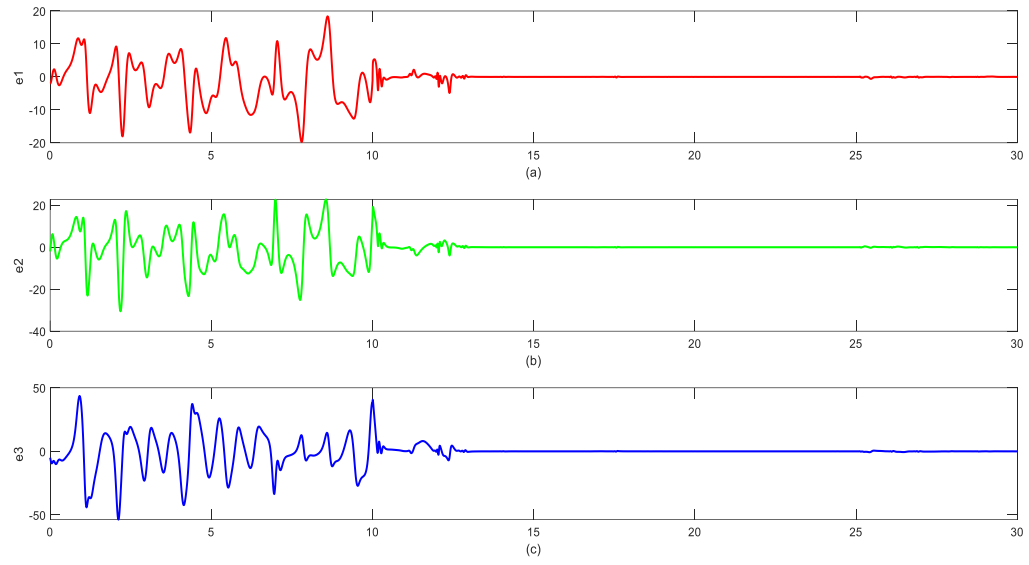


Fig. 4.11: The retort of error signs for synchronization when the passive controller is activated $t=10$ (a) e_1 signs, (b) e_2 signs, (c) e_3 signs

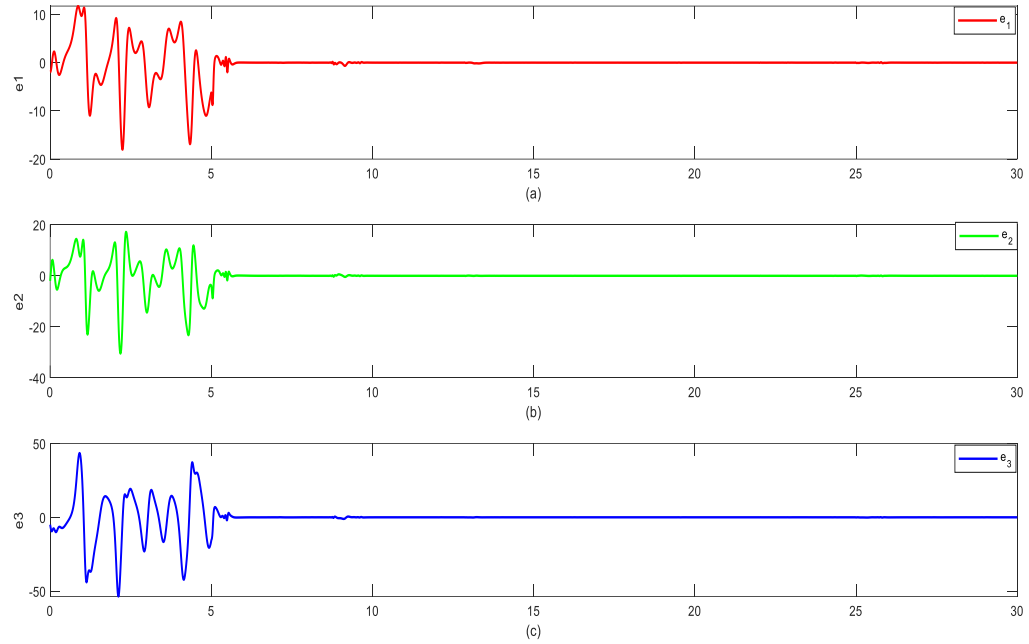


Fig.4.12: The retort of error signs for synchronization when the passive controller is started $t=5$ (a) e_1 signs, (b) e_2 signs, (c) e_3 signs.

The foremost empirical of this exploration is to investigate the synchronization of chaos in a nonlinear finance system using the Liu dynamical system. A low-dimensional financial system can adjust to the global macroeconomic system through synchronization. Rapid fluctuations in factors like price and interest rates are the primary causes of changes in demand and volume. Rapid changes in factors such as price and interest rates are the main drivers behind shifts in demand and volume. These cause nonlinearity within a system. Alignment with the global financial system offers advantages for economic growth by achieving uniform interest rates, investment demand, and price levels. Additionally, it can help mitigate uneven economic risks. Two sliding mode controllers and one passive controller have been created using the theories of sliding mode and passive control to bring two identical chaotic financial systems into synchrony. All theoretical analyses of the proposed methods of control successfully synchronize the two chaotic financial systems, based on numerical simulations. In each case depicted in Figs. 4.5–4.12, sliding mode controllers are more effective than passive controllers at controlling the synchronization of chaotic finance systems; thus, the sliding mode approach is more suitable. The benefit of the passive control approach is that it allows chaotic financial systems to be synchronized with a single controller, making implementation easier. The passive control merely modifies the interest rate, whereas the sliding mode control achieves synchronization by modifying both the interest rate and price index. The study helps researchers to choose the best technique depending on system restrictions by highlighting the trade-offs between the two approaches.

Chapter 5: CHAOS REGULATION AND SYNCHRONIZATION OF MODIFIED LORENZ SYSTEMS THROUGH SLIDING MODE CONTROL

5.1 Introduction

Dynamical systems and their chaotic nature have drawn substantial interest from a variety of disciplines, including biology, engineering, and physics. The fundamental dynamics of chaotic systems must be understood and harnessed using chaos control and synchronization strategies. Chaos, as a complex and nonlinear dynamic process, is prevalent across various fields. Systems like levitation by magnets, windmill surface models, radar connectivity [109], algorithms for encryption, digital circuits, quarter automobile vehicle models, traveler ground vehicle oscillation systems, two-dimensional chemical reaction-diffusion models, Aihara neuron wireless networks, field configurable Gate Arrays, portable robotics, economic modeling, and even mental health are among the many systems in which it is important [110].

Since the creation of the Lorenz system, chaos has been one of the most significant topics in nonlinear science, and it has been thoroughly studied. Chaos theory was developed in the 1960s by the American meteorologist Lorenz [1] who attempted to make theoretical predictions of weather forecasts based on the theory of fluid dynamics and found that subtle differences in initial states caused large and unpredictable changes in weather models. Since the first publication of the Lorenz model, chaotic behavior has been of interest as one of the important topics in the analysis of nonlinear systems, and it is generally known to be sensitive to small changes in initial conditions. The most representative chaos systems known so far are the Chen system, the Lü system [111], the Rössler system, and the Lü-Chen system [112], all of which are composed of three-dimensional variables and are characterized by sensitive reactions to the initial state. Chaotic systems are dynamic and highly reactive to starting conditions, indicating behaviors that look complex and unpredictable. Despite their deterministic nature, these systems remain unpredictable. In statistical systems, chaos is defined by its random-like behavior; in predictable systems, on the other hand, it exhibits unique behavior. Since it is typically challenging to predict with precision how a chaotic system will behave in the future, chaos could cause systems to become unstable, perform worse, or even collapse, to the point where chaos is viewed as undesirable and often needs

to be removed [113]. Numerous engineering systems have been known to exhibit chaos. The application of chaos theory and chaotic properties can be helpful in many engineering fields, such as digital messaging, reliable communication, electrical devices and power quality, biological structures, chemical response analysis and design, and data processing. Current approaches to chaos control include a range of strategies intended to control and stabilize chaotic behavior in dynamical systems. Lyapunov control [114] and other nonlinear control strategies are one of the notable methods. Lyapunov control is based on Lyapunov stability theory, which is concerned with establishing and ensuring the stability of nonlinear systems. A Lyapunov function is used in Lyapunov control to examine and regulate the dynamics of a system [115]. Electrical circuits, robotics, chaotic systems, and other engineering applications frequently employ Lyapunov control, which is especially well-suited for nonlinear systems [116]. Nonlinear control methods encompass various approaches that control nonlinear functions to regulate and stabilize chaotic systems. This category includes feedback control, sliding mode control, adaptive control and others. In recent years, many researchers have shown a growing interest in the control of chaotic systems. The author implemented a sliding mode controller to suppress chaos in the Liu system, a recently developed uncertain chaotic dynamical system. In [117], a robust adaptive sliding mode controller is employed to eliminate chaos from a new class of chaotic systems. Sliding mode control is a straightforward method for managing chaos that has been successfully used in uncertain systems [118] and control nonlinear systems [119]. Russian scientists developed variable structure control, also referred to as sliding mode control, in the early 1960s and later documented it in Utkin's monograph in the 1970s [120]. Since that time, due to its simplicity and adaptability to system unpredictability and changes SMC has been extensively studied and successfully applied to numerous real-world control problems [121]. SMC's sliding motion is its most appealing feature. During the sliding mode, the system's dynamic motion is essentially limited to a particular area of the whole state space.

The creation of a sliding surface, which is usually described in terms of the system's state variables, is the fundamental component of SMC. A hyperplane, or flat surface, is selected as the sliding surface, a mathematical construct that shows the planned system behavior in the state space. The system's state will be motivated to remain on this sliding surface by the control law's design. Two modes of operation are available to SMC's control law. The control

law is designed to guide the system toward the sliding surface when it is not yet in the "reaching phase" of the system's state. As soon as the condition reaches the sliding surface, the control law changes to a new mode (the "sliding phase") intended to keep the system on the surface. When the control signal suddenly changes between two values, it is a phenomenon linked to SMC. In certain systems, this high-frequency switching may result in undesired mechanical wear. In real applications, chattering is reduced by methods like reaching law smoothing and boundary layer design. The capacity of sliding mode control to withstand model uncertainties as well as external disturbances is one of its main benefits. This system is ideal for situations where accurate modeling is difficult because it stays on the sliding surface even when uncertainties are present. The goal of stability analysis in SMC is to make sure that the system stays stable as it slides across the surface. To ensure accurate and stable operating, the sliding mode controller aims to move the system towards the sliding surface while remaining it there. The principal advantages of Sliding Mode Control (SMC), which include its robustness to uncertainties, adaptability to nonlinear systems, and applicability in challenging control scenarios works well for managing nonlinear systems which are frequently found in real-world situations. Since SMC doesn't rely on linearization like linear control methods do, it can handle a variety of nonlinear dynamics. It is a flexible option for a range of applications because it can regulate systems with intricate and time-varying behaviors. When the system's operating conditions or disturbances change, SMC responds quickly. The control law quickly changes to keep the system on the sliding surface. SMC is especially well-suited for applications like robotics and aerospace where rapid disturbance rejection is crucial because of this property. In terms of control law formulation and implementation, SMC is known for its simplicity. This feature can lessen processing needs and simplify the design of control systems, making them suitable for real-time applications is especially helpful in situations where obtaining a precise mathematical model of the system is difficult or impractical because it can be implemented without one. A few real-world applications benefit from this model-free feature. Sliding mode control, or SMC, has been effectively applied to the management of uncertain and nonlinear systems in several fields. These are a few intriguing examples of past applications: DC-DC converters are controlled by aircraft control systems to ensure strong and dependable performance even in the face of unknown load situations and changing parameters; nonlinearities and

uncertainties in the robot's environment are managed by controlling the motion of mobile robots. These examples show how to provide resilience against external disturbances and uncertainties in aerodynamic parameters in aircraft control systems offers robustness against variations in wind speed and turbine [122] parameters and has been utilized in wind turbine control to optimize power extraction. To ensure strong synchronization in the face of noise and uncertainties, SMC has been used in chaos-based communication systems [123].

The synchronization problem discussed in this chapter has received a lot of attention since it was first proposed by Pecora and Carroll, and it has been applied to secure communication and system identification, and the synchronization methods include nonlinear feedback [124], active control [125], back stepping [126], and sliding mode control (SMC) [127].

In this chapter, we use the drive-response technique for synchronizing chaos systems. This technique consists of a drive system paired with a state error variable and a responsive stem. There are some interesting previous application examples: The goal of synchronization is to follow the drive system's state and apply control inputs to the response system based on errors between the drive and response frameworks. On the other hand, as a control technique, an active sliding mode controller is applied. To achieve this, choose a suitable dynamic controller for the sliding mode controller and use it to synchronize it with the sliding mode controller. In general, sliding mode control decreases the chattering phenomenon caused by discrete control.

Particularly, it rapidly reaches the sliding plane by reducing the chattering phenomenon caused by discontinuous control in general sliding mode control. The proposed approach is then used to run a computer simulation, which serves to validate its applicability. We noticed an efficient method to control chaotic systems via motivations. This article uses sliding mode controllers to establish global asymptotical synchronization between the chaotic systems and offers worldwide synchronization between two identical chaotic systems.

Our focus is to examine how sliding mode controllers can be used to control and synchronize the nonlinear dynamical system, or MLS. The controller is developed with uncertainty in the system and torque disturbance in view. Using MATLAB, solve the nonlinear tracking problem with continuously switching topologies and present the numerical results to illustrate the usefulness of the control law.

The underlying theoretical basis for our work is detailed in Section 5.1. We'll start our work by organizing it in the following manner: In Section 5.2, MLS's dynamical system is examined. In Section 5.2.1, sliding mode control is employed to achieve synchronization of MLS systems. In Section 5.3, the results of the numerical simulations are presented.

5.2. A Three-Dimensional Modified Lorenz System (MLS)

A three-dimensional Modified Lorenz System (MLS) can be described as [128]

$$\begin{aligned}\dot{x} &= a_1(b_1y - x); \\ \dot{y} &= c_1x - xz; \\ \dot{z} &= xy - b_1z;\end{aligned}\tag{5.1}$$

In this case, the condition variables are x, y, z and the inputs are a_1, b_1, c_1 . In order to verify unpredictable behavior for conditions and determine the parameter values, first introduce the values as follows.: $x_1(0) = 2.3, y_1(0) = -1.3, z_1(0) = 1$ and $x_2(0) = 24, y_2(0) = 20, z_2(0) = 28$, by taking parameters $a_1 = 10, b_1 = 4.845, c_1 = 20$ and then the system Equation (5.1) is chaotic. Phase portraits of 2D and 3D are displayed in Fig.5.3 and 5.4, respectively, while time series of the complex system (5.1) for various initial condition values are shown in Fig.5.1 and 5.2.

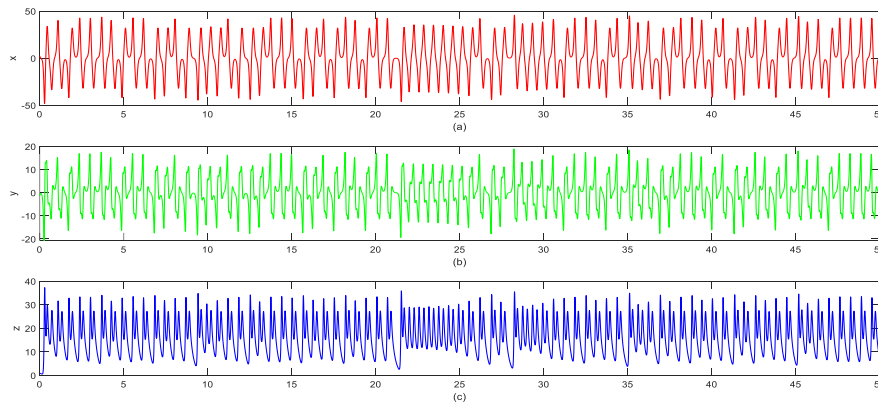


Fig.5.1: Time series plots for (a) x signs, (b) y signs, (c) z signs with initial conditions $x_1(0) = 2.3, y_1(0) = -1.3, z_1(0) = 1$ respectively.

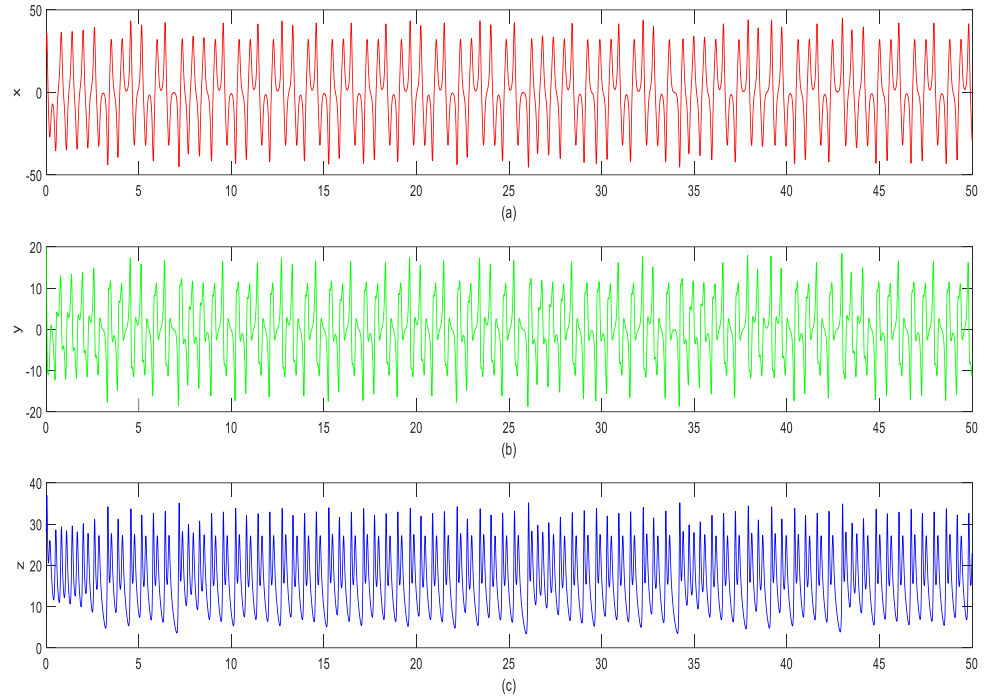


Fig.5.2: Time series plots for (a) x signs, (b) y signs, (c) z signs with initial conditions $x_2(0) = 20, y_2(0) = 24, z_2(0) = 28$ respectively.

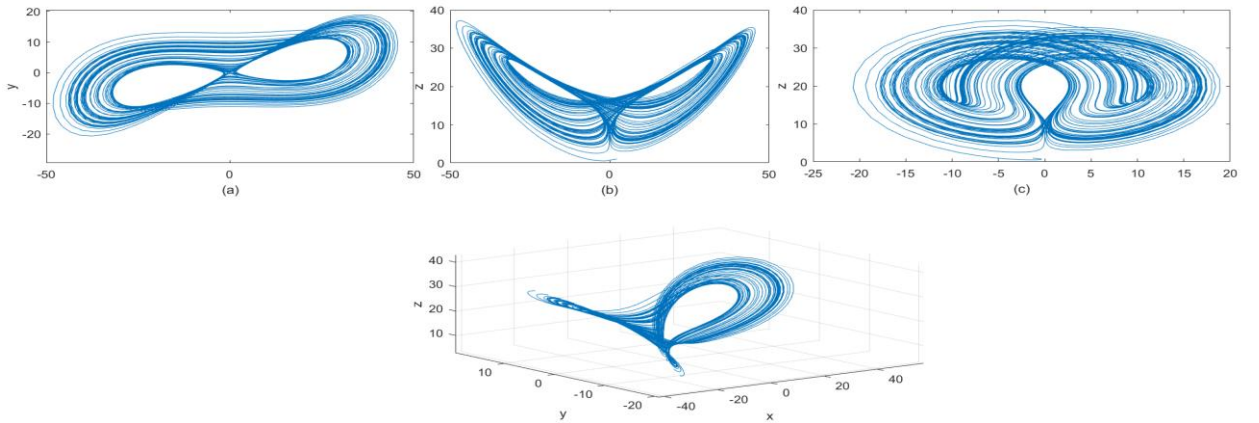


Fig.5.3: Phase portraits of 2D are in (a) xy plane, (b) xz plane, (c) yz plane and 3D phase portrait in the system with initial conditions: $x_1(0) = 2.3, y_1(0) = -1.3, z_1(0) = 1$, respectively.

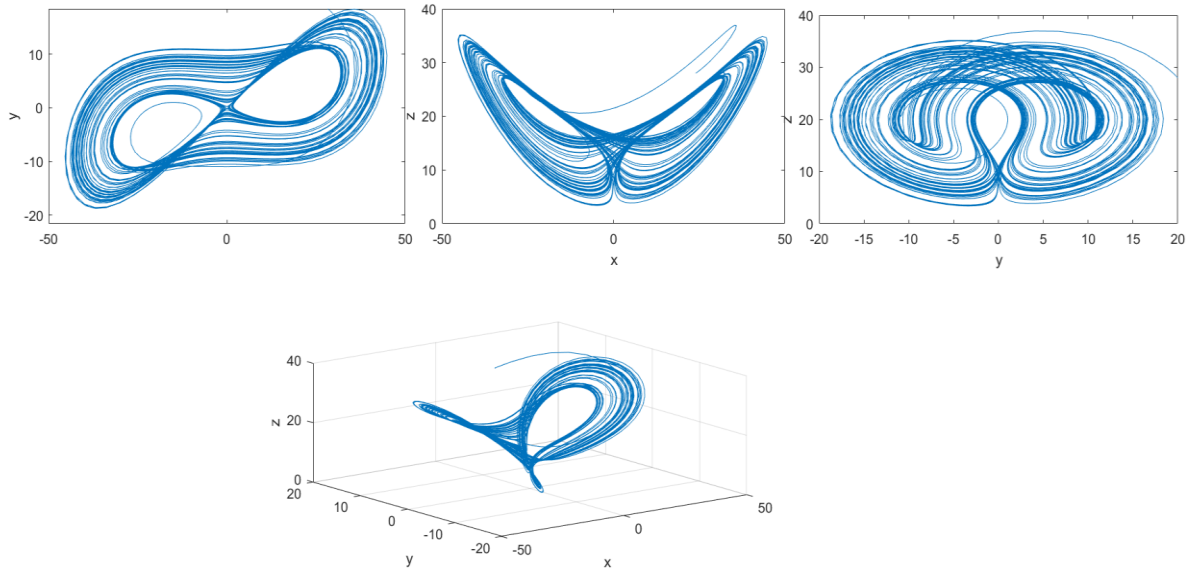


Fig.5.4: Phase portraits of 2D are in (a) xy plane, (b) xz plane, (c) yz plane and 3D phase portrait in the system with initial conditions $x_2(0) = 24, y_2(0) = 20, z_2(0) = 28$, respectively.

5.2.1 Apply of Sliding Mode Control (SMC) Method to MLS

The parameters a , b , and c are taken within a range, ensuring that the system (5.1) will exhibit unpredictable behavior. Two chaotic MLS systems, where the drive system controls the response system, are believed to have been taken after synchronization was observed. Subscript 1 represents the drive system, and subscript 2 represents the response system.

$$\begin{aligned}\dot{x}_1 &= a_1(b_1y_1 - x_1) \\ \dot{y}_1 &= c_1x_1 - x_1z_1 \\ \dot{z}_1 &= x_1y_1 - b_1z_1\end{aligned}\tag{5.2}$$

and the response system is described as

$$\begin{aligned}
\dot{x}_2 &= a_1(b_1y_2 - x_2) + u_1(t) \\
\dot{y}_2 &= c_1x_2 - x_2z_2 + u_2(t) \\
\dot{z}_1 &= x_1y_2 - b_1z_2 + u_3(t)
\end{aligned} \tag{5.3}$$

where $u_1(t)$, $u_2(t)$, $u_3(t)$ in Eq. (5.3) are to be determined. The control functions are obtained by subtracting the drive system from the response system for synchronization. The state errors e_1 , e_2 , and e_3 are to be controlled and the controlling system (5.2) are expressed as

$$\begin{aligned}
e_1 &= x_2 - x_1 \\
e_2 &= y_2 - y_1 \\
e_3 &= z_2 - z_1
\end{aligned} \tag{5.4}$$

Thus, the patterns of errors become

$$\begin{aligned}
\dot{e}_1 &= a_1b_1(y_2 - y_1) - a_1(x_2 - x_1) + u_1(t) \\
\dot{e}_2 &= c_1e_1 - x_2z_2 + x_1z_1 + u_2(t) \\
\dot{e}_3 &= x_2y_2 - x_1y_1 - b_1e_3 + u_3(t)
\end{aligned} \tag{5.5}$$

In matrix notation, we can normalize the error behavior (5.5) to:

$$\dot{e} = Ae + \eta(x, y) + u \tag{5.6}$$

where,

$$A = \begin{bmatrix} -a_1 & a_1b_1 & 0 \\ c_1 & 0 & 0 \\ 0 & 0 & -b_1 \end{bmatrix}, \quad \eta(x, y) = \begin{bmatrix} 0 \\ -x_2z_2 + x_1z_1 \\ x_2y_2 - x_1y_1 \end{bmatrix}$$

$$u = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} \tag{5.7}$$

The meaning of the control signal u based on the sliding mode control methodology is as follows[130]:

$$u(t) = -\eta(x, y) + Bv(t) \quad (5.8)$$

where B is a matrix and v is a control signal. In order to make (A, B) controllable, B was selected.

Therefore, B is taken as,

$$B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad (5.9)$$

Substituting (5.8) into (5.6) the error behavior simplifies to,

$$\dot{e} = Ae + Bv \quad (5.10)$$

Which is a linear time-invariant control system with single input v .

By choosing a suitable sliding mode control, the original global chaos synchronization problem can be transformed into an equivalent problem of stabilizing the zero-solution $e = 0$ of the system in equation (5.10). The variable is defined within the sliding mode control

$$s(e) = Ce = c_1 e_1 + c_2 e_2 + \dots + c_n e_n. \quad (5.11)$$

as, $C = [c_1 \ c_2 \ \dots \ c_n]$ is a constant vector needs to be determined.

Now reduce the system (5.10) motion in the sliding mode control to the sliding manifold denoted by, $s = \{x \in R^n, |s(e)| = 0\}$

which needs to remain unchanged under the influence of the error dynamics described by equation (5.10). when the sliding manifold s , the system (5.10) satisfies the following conditions:

$$s(e) = 0. \quad (5.12)$$

which is the defining equation for the manifold s and,

$$\dot{s}(e) = 0. \quad (5.13)$$

In the chaotic case, the parameter values, $a_1 = 10$, $b_1 = 4.845$, $c_1 = 20$

The sliding mode variable is selected as,

$$S=Ce = [2 \ 1 \ -1]e = 2e_1 + e_2 - e_3.$$

which ensures that the sliding mode state equation is asymptotically stable.

Select the sliding mode gain as follows, $K=5$ and $q=0.1$

A high value of k can lead to chattering, while a suitable value of q is selected to both accelerate the time it takes to reach the sliding manifold and minimize system chattering. Using (5.10) and (5.11) the equation (5.13) can be rewritten as

$$\dot{s}(e) = C(Ae + Bv) = 0. \quad (5.14)$$

From the property of the sliding mode control theory,

$$v(t) = -(CB)^{-1}[C(KI + A)e + q\text{sign}(s)]. \quad (5.15)$$

where C is chosen such that $CB \neq 0$

Now, the $v(t)$ control signal becomes,

$$v(t) = -[-10e_1 + 96.9e_2 - 0.155e_3 + 0.1\text{sign}(s)] \quad (5.16)$$

As a result, the necessary sliding mode controller is acquired as

$$u = -\eta(x, y) + Bv \quad (5.17)$$

where $\eta(x, y)$, B and $v(t)$ are defined as in the equation (5.7), (5.9) and (5.16).

Then, the required sliding mode control signal is obtained as Eq. (5.8) where $\eta(e)$ and B are described as in Eq. (5.7) and (5.9), respectively:

$$u_1(t) = 0$$

$$u_2(t) = x_2z_2 - x_1z_1 + v(t), \quad (5.18)$$

$$u_3(t) = -x_2y_2 + x_1y_1 + v(t)$$

Another sliding mode variable is selected as,

$$S = Ce = [2 \ 1 \ -2]e = 2e_1 + e_2 - 2e_3.$$

Now, the $v(t)$ control signal becomes,

$$v(t) = -[10e_1 + 101.9e_2 - 0.31e_3 + 0.1\text{sign}(s)] \quad (5.19)$$

As a result, the necessary sliding mode controller is acquired as

$$u = -\eta(x, y) + Bv$$

Equations (5.16) and (5.19) are now compared with the present equation to analyze their structural similarities and differences.

5.3 Results and Discussions

The MLS systems (5.2) and (5.3) are solved in this section of the numerical simulations using the sliding mode controller u that comes with (5.17) and MATLAB's fourth-order Runge-Kutta method with a fixed step size of 0.001

In the disordered case, the parameter values are, $a_1 = 10, b_1 = 4.845, c_1 = 20$

The sliding mode increases are chosen as, $k=5$ and $q=0.1$

The initial values of the drive system (5.2) are taken as

$$x_1(0) = 2.3, y_1(0) = -1.3, z_1(0) = 1$$

and the initial values of the response system (5.3) are taken as

$$x_2(0) = 24, y_2(0) = 20 \text{ and } z_2(0) = 28$$

In Fig.5.5, the sliding mode controllers have achieved synchronization of chaotic system at $t = 10$, The initial conditions are defined for both the drive and response systems. The system is simulated with control inputs for $t \geq 10$ and Fig.5.6, the system is simulated with

control inputs for $t \geq 5$, the observed simulation results for the synchronization of two identical chaotic MLS systems are shown in Fig.5.5 and 5.6.

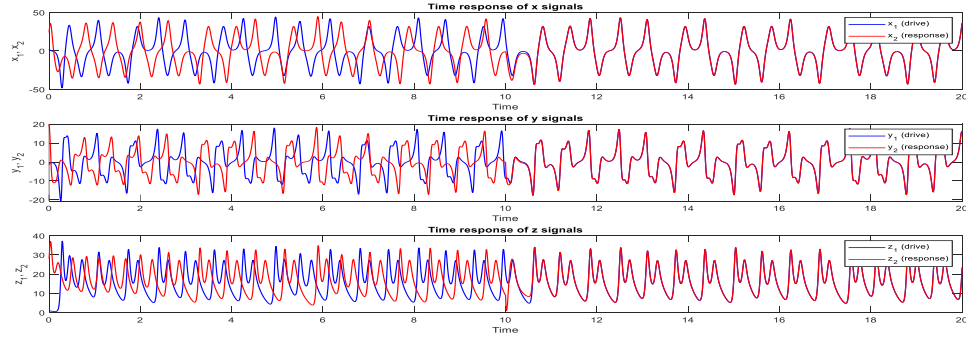


Fig.5.5: The time response of states for synchronization of MLS systems with the controllers are activated at $t = 10$ (a) x signals, (b) y signals, (c) z signals

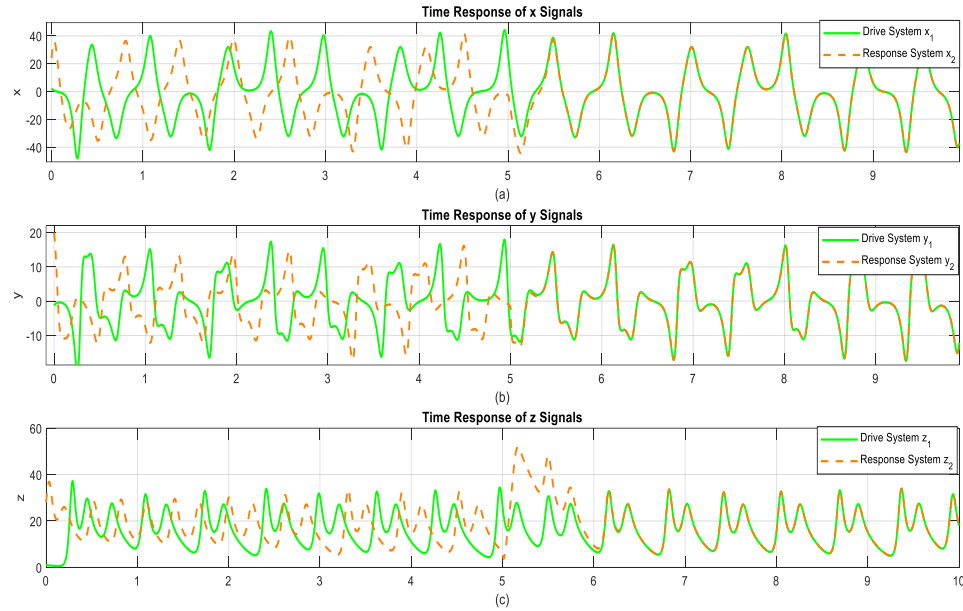


Fig. 5.6: The time response of states for synchronization of MLS systems with the controllers are activated at $t = 5$ (a) x signals, (b) y signals, (c) z signals.

In Fig.5.7 and 5.8, The error signals e_1 , e_2 and e_3 are plotted to observe the convergence and synchronization between the drive and response systems.

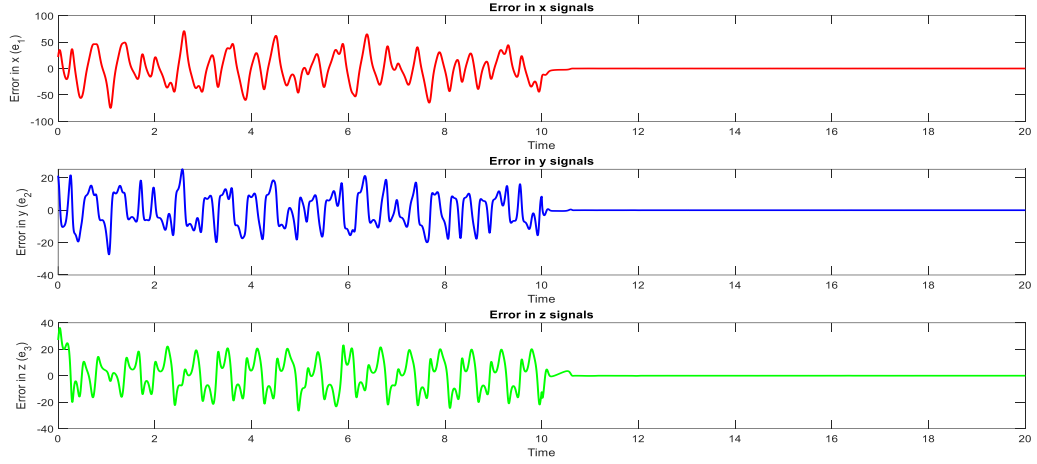


Fig.5.7: Dynamics of the synchronization error (e_1, e_2, e_3) with the controller activated at $t \geq 10$.

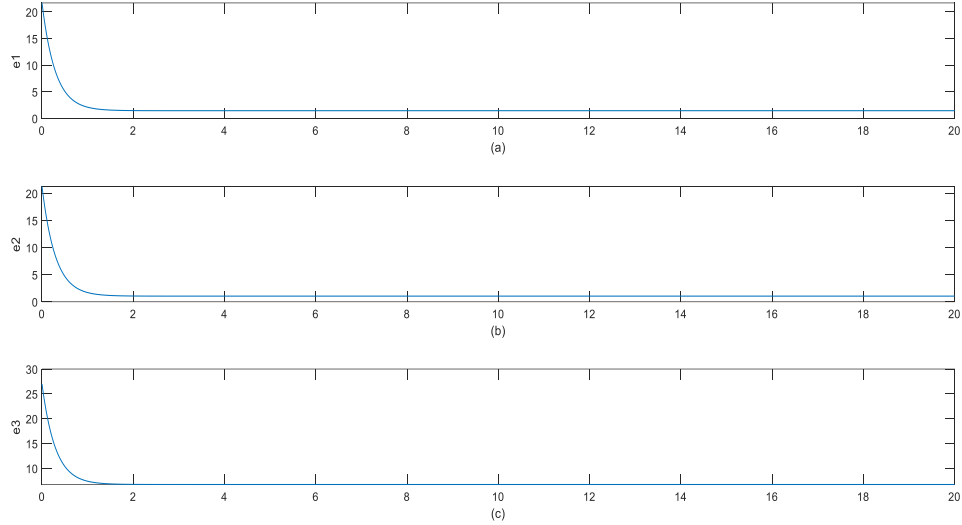


Fig. 5.8: Dynamics of the synchronization error (e_1, e_2, e_3) with the controller activated at $t = 20$

In Fig.5.9 and 5.10, 3D portrait of the state variables of the drive and response system with the controller activated at $t = 0$. Error terms e_1, e_2, e_3 are computed and used to calculate the control inputs u_1, u_2, u_3 . The function $v(t)$ is defined to apply the specified control strategy.

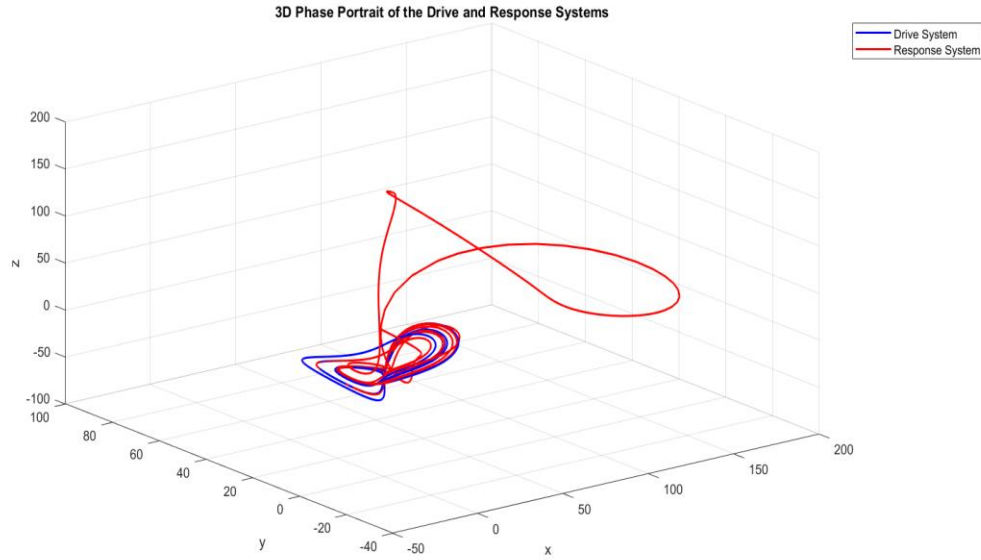


Fig.5.9: 3D portrait of the state variables of the drive and response system with the controller activated at $t = 0$

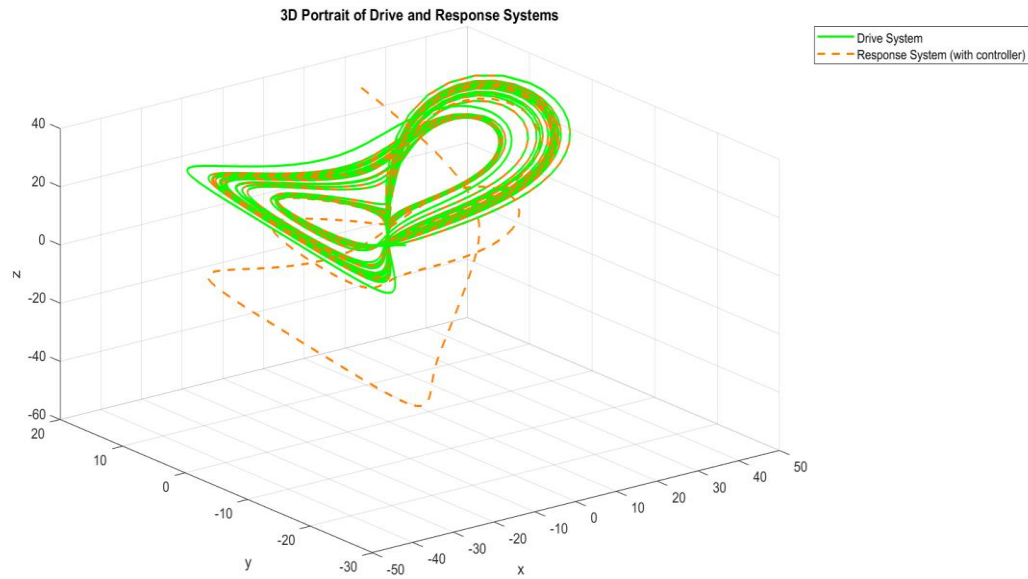


Fig. 5.10: 3D portrait of the state variables of the drive and response system with the controller activated at $t = 0$

In fig.5.11 and 5.12, when the sliding mode controllers are activated at $t \geq 1.5$ and the error controllers are activated at $t = 2$

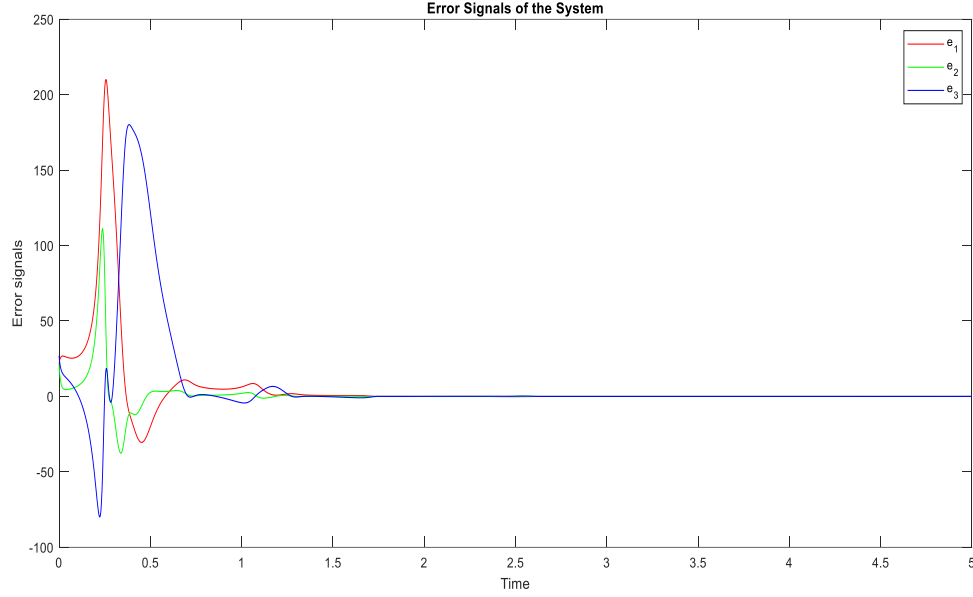


Fig. 5.11: The time response of states for synchronization of MLS systems with the controllers are activated at $t = 2$ (a) e_1 signals, (b) e_2 signals, (c) e_3 signals

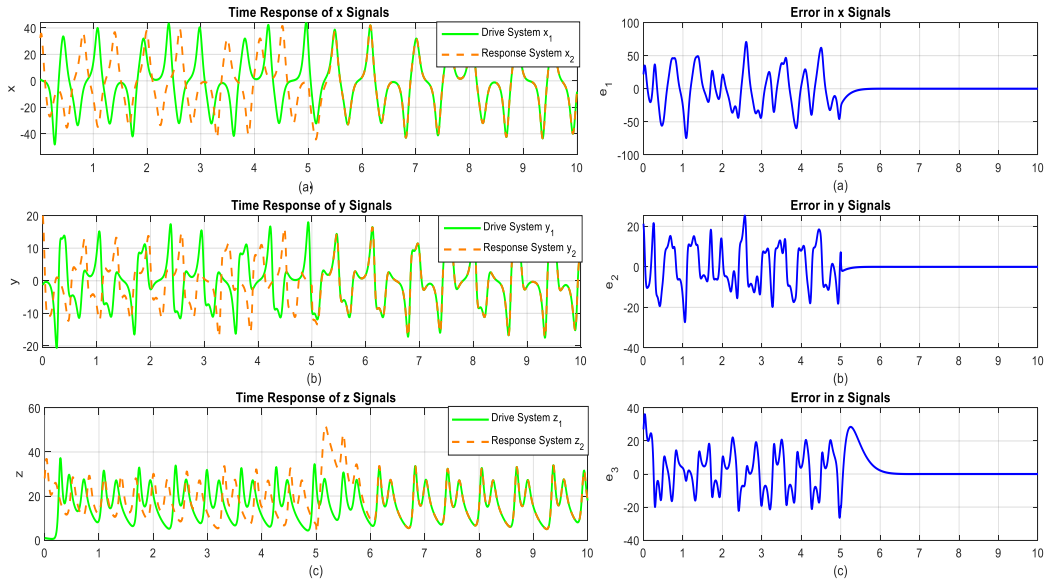


Fig. 5.12: The time response of states for synchronization of MLS systems with the controllers are activated at $t > 5$ and an error are activated at $t = 5$ (a) x signals, (b) y signals, (c) z signals

In Fig.5.13, the controllers are activated at $t \geq 1$ and Fig.5.14, the controllers are activated at $t \geq 10$.

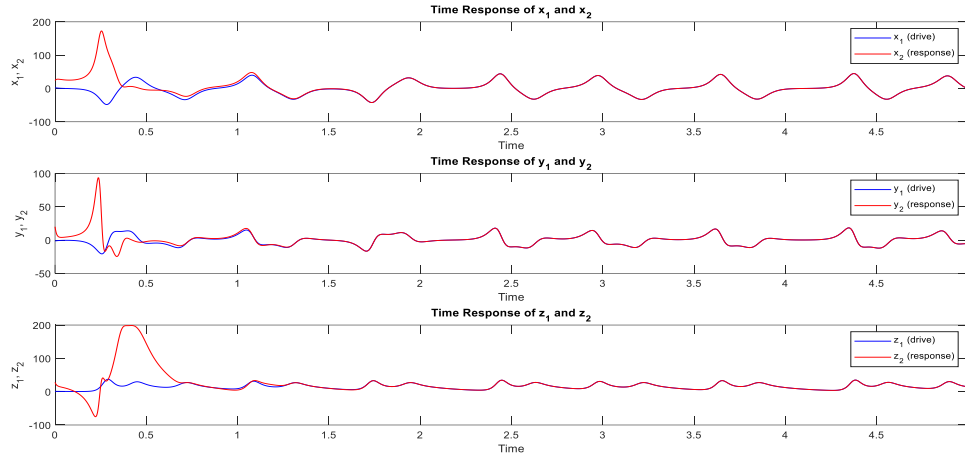


Fig. 5.13: The time response of states for synchronization of MLS systems with the controllers are activated at $t = 5$ (a) x signals, (b) y signals, (c) z signals

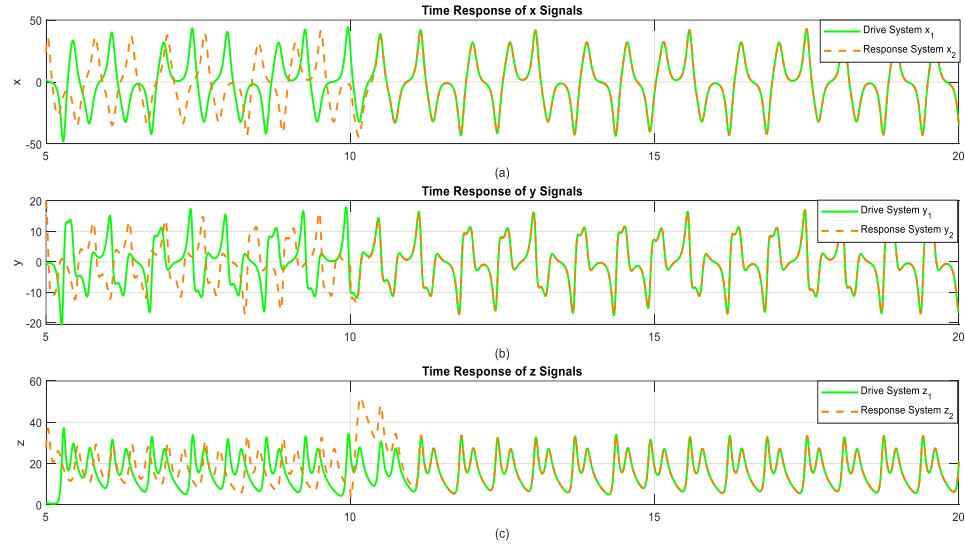


Fig. 5.14: The time response of states for synchronization of MLS systems with the controllers are activated at $t \geq 10$ (a) x signals, (b) y signals, (c) z signals

The use of intelligent computers is expanding in new areas of science, and it is feasible to apply advanced control algorithms. The creation of developed control systems, particularly sliding mode control, makes sense because of its benefits and ease of application for both linear and non-linear systems. These benefits include its recommendations for nonlinear

control, its insensitivity to corresponding uncertainties, its effective order sliding mode equations, and its zero-error convergence in a closed-loop system. This article examines the non-linear dynamics conditions for synchronization and chaos control in a modified several-linear system model. The SMC method is a novel technique that has been used in this system to achieve synchronization or chaos control. Furthermore, it has been shown that the sliding mode state equation exhibits asymptotically stable behavior under specific choices of the sliding surface vector. Our method has drawbacks, despite its benefits. In this case, the impact on our chaotic systems has not been considered. It is commonly known that noises from external circumstances may severely interfere with certain dynamical systems. Therefore, it is essential to control and synchronize chaotic output systems with outside disturbances. Our top priority for research will be this topic. As a consequence, the results obtained show that the system is synchronized. All the analytical findings have been confirmed using numerical simulations.

Chapter 6: CONCLUSION

The journey of this research began with a comprehensive literature review on chaos theory and synchronization. Since the pioneering work of Pecora and Carroll, chaotic synchronization has evolved as a critical technique with wide-ranging applications in secure communications, system identification, neural networks, cryptography, robotics, and power systems. Several control strategies have been developed over the decades—such as nonlinear feedback, adaptive control, backstepping, and active control—but sliding mode control continues to attract attention due to its distinct advantages.

Our research successfully addressed several core challenges associated with SMC, especially the issue of chattering—the undesirable high-frequency switching in control signals that can cause mechanical wear in physical implementations. To mitigate this, we designed a smooth dynamic sliding surface and incorporated a boundary layer approach and continuous approximation methods within the reaching law, thereby preserving the robustness of SMC while reducing chattering.

The drive-response synchronization framework adopted in this study is particularly effective in chaotic systems. The drive system, acting as the master, evolves based on its inherent chaotic dynamics, while the response system receives control signals derived from the error states. These error signals reflect the divergence between the two systems' state variables. By designing suitable control inputs based on these errors, the response system is driven to track the drive system's behavior, ultimately achieving synchronization.

The practical implications of this research are noteworthy. Sliding mode control offers an edge over other synchronization methods due to its insensitivity to model inaccuracies, its capacity to reject disturbances, and its relatively simple design framework. In engineering domains where reliability and robustness are critical—such as aerospace systems, electric drives, and automation—the proposed method can provide stable synchronization even when the exact mathematical model of the system is difficult to obtain.

Furthermore, the ability to synchronize chaotic systems opens avenues in secure communication, where synchronization can be used to encrypt and decrypt information transmitted through chaotic signals, ensuring high-level security.

Despite the success and robustness of the proposed strategy, this study also has some limitations that warrant further exploration:

- The current simulations were conducted under ideal noise-free conditions. Real systems are often influenced by measurement noise, actuator delays, and environmental factors that can degrade performance.
- The controller parameters were pre-defined and not adaptively tuned during operation. In practice, adaptive tuning may be necessary to cope with changing dynamics.
- Although chattering was reduced, the smooth approximation may sacrifice some robustness, particularly in the presence of large disturbances.

Overall, this thesis has presented a comprehensive study on synchronizing chaotic systems using an enhanced sliding mode control approach. Through mathematical formulation, controller design, and simulation validation, we have shown that the proposed strategy ensures global asymptotic synchronization of chaotic dynamics with strong robustness properties.

The dynamic sliding surface and active control design not only ensure the convergence of synchronization errors but also effectively reduce chattering—one of the major drawbacks of classical SMC methods. The controller’s simple structure, low computational demand, and high efficiency make it a strong candidate for real-time implementation in various fields where chaotic systems are prevalent.

This research contributes meaningfully to the body of knowledge on chaos control and synchronization and lays a solid foundation for future advancements in control engineering, especially in the synchronization of complex nonlinear systems. We hope that this work will inspire further exploration into robust, adaptive, and intelligent control strategies for managing chaos in the real-world systems.

Suggestions For Further Study

The results of this study open the door for several promising directions of future research:

1. Many real-world systems are better modeled using fractional-order differential equations. Extending this control approach to such systems could significantly broaden its application.
2. Combining SMC with adaptive control, fuzzy logic, or neural networks could enhance the system's ability to handle time-varying uncertainties and nonlinearities dynamically.
3. A practical implementation of the proposed control scheme in a hardware-in-the-loop environment would offer valuable insights into real-time performance and practical feasibility.
4. Synchronization of networks of chaotic systems (multi-agent systems) under sliding mode frameworks could have significant applications in distributed control, sensor networks, and swarm robotics.
5. In many practical cases, not all states are measurable. Developing observer-based sliding mode controllers can help achieve synchronization using only partial system information.
6. Addressing synchronization in chaotic systems affected by noise and random disturbances could make the method more applicable to communication and signal processing fields.

Computer simulations are conducted to validate the effectiveness of the proposed synchronization strategy. Our findings demonstrate that sliding mode control offers an efficient method for achieving global asymptotic synchronization between chaotic systems.

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